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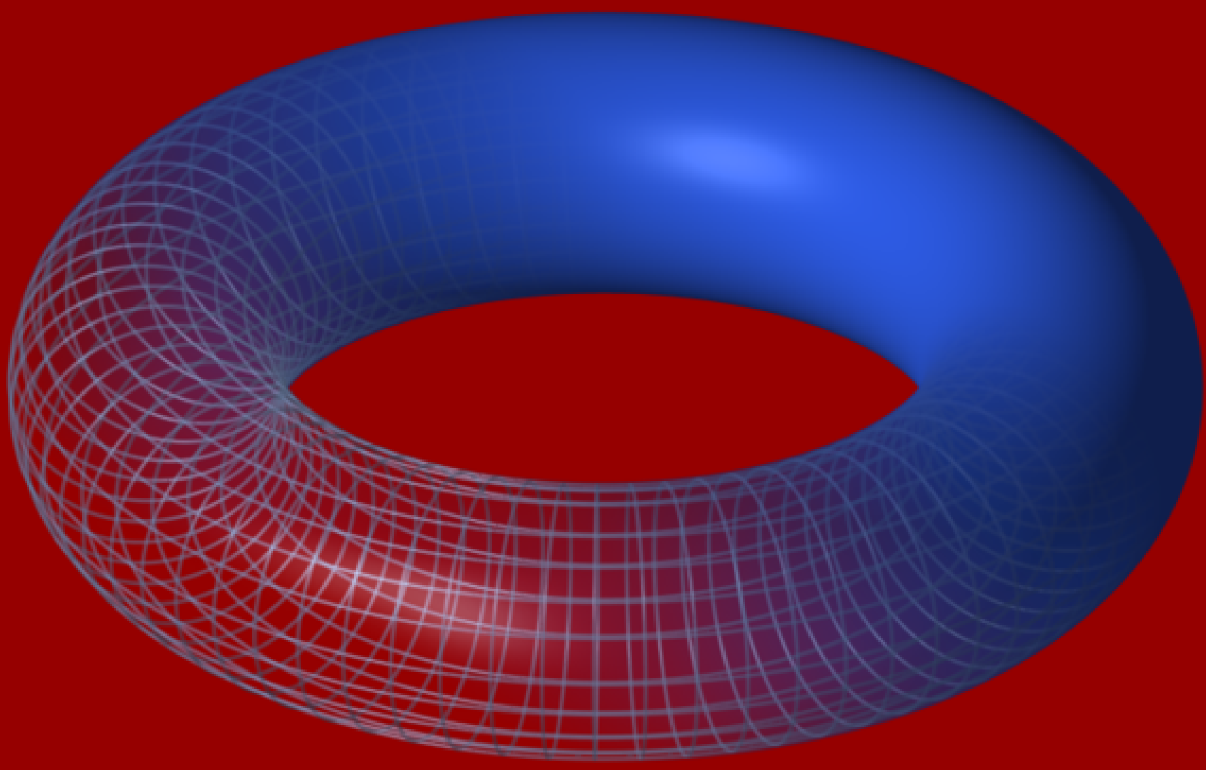

TOLASO NETWORK
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The JoM Journal

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The JoM Journal is an electronic mathematical journal which aims at giving the chance to the readers, and the editor himself, to work on interesting mathematical problems or find information about various mathematical topics. The problems presented here are basically a collection of the problems posted on the JoM Blog (hosted at math.tolaso.com.gr) . The level of the topics is undergraduate and beyond. However, there is a section dedicated to inequalities and general mathematics sometimes including mathematical competitions. The JoM journal is consisted of 6 parts:

- Algebra
- Calculus
- Real Analysis
- Inequalities
- JoM ... proposes
- JoM ... study

The JoM ... proposes column contains problems that extend the ideas already seen in the previous 4 columns. The JoM study, on the other hand, studies several mathematical concepts. Examples are included whenever necessary. At the end of this part the reader will find problems to exercise himself.

If you want to submit an article at the JoM ... study please contact the author at tolaso@tolaso.com.gr.

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PART 1

Algebra

(1) Let $n \in \mathbb{N}$ and $1 \le k \le n$. Evaluate the determinant

$$\mathcal{D} = \begin{vmatrix} \binom{n}{0} & \binom{n}{1} & \cdots & \binom{n}{k} \\ \binom{n+1}{0} & \binom{n+1}{1} & \cdots & \binom{n+1}{k} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{n+k}{0} & \binom{n+k}{1} & \cdots & \binom{n+k}{k} \end{vmatrix}$$

Solution. *Let $r_1, \ldots, r_{k+1}$ be the rows of the matrix. For $m = k, k-1, \ldots, 1$ with this order , I change the r_{m+1} at the row*

$$\sum_{\ell=0}^{m} \binom{m}{l} (-1)^{m-\ell} r_{\ell+1}$$

The determinant stays the same. After all the changes at the $m+1$ row and $r+1$ column we get

$$a_{m+1,r+1} = \sum_{\ell=0}^{m} \binom{m}{\ell} (-1)^{m-\ell} \binom{n+\ell}{r} = \sum_{\ell=0}^{m} \binom{m}{\ell} (-1)^{m-\ell} p_{n,r}(\ell)$$

where $p_{n,r}$ is a polynomial of order r with the greatest coefficient being 1. We claim that $a_{m+1,r+1}$ equals 0 if $r < m$ and 1 if $r{=}m$. Suffice to prove that

$$\sum_{\ell=0}^{m} \binom{m}{\ell} (-1)^{m-\ell} \ell^s$$

equals to 0 if $s < m$ and 1 if $s = m$. This follows from the inclusion - exclusion principal since the last sum is the number of partitions of $\{1, 2, \ldots, s\}$ into m non empty subsets.

After all these we get that the new matrix is upper triangular with 1's in its main diagonal. Thus, $\mathcal{D} = 1$.

♦

(2) Let $A \in \mathbb{C}^{n \times n}$ be similar to A^2. Does $A = A^2$ hold?

Solution. *No! Take $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ then $A^2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. The matrices are similar but not equal.*

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3727`.

(3) For any two symmetric $n \times n$ matrices A and B be their eigenvalues be ordered from largest to smallest . Prove that for eigenvalues $|\lambda_k^A - \lambda_k^B| \le \|A - B\|$ for $1 \le k \le n$. where λ_k^A, λ_k^B are respective eigenvalues of A and B.

Solution. *This follows from the following characterisation of the eigenvalues of symmetric matrices.*

$$\lambda_k(A) = \min_{\dim(U)=n-k+1} \max_{0 \ne x \in U} \frac{x^T A x}{\|x\|_2}$$

Here the minimum is taken over all subspaces U of $\mathbb{R}^n$ of dimension $n-k+1$. (I chose to write $\lambda_K(A)$ rather than λ_k^A.) The proof of this uses the fact that symmetric matrices have an orthonormal basis of eigenvectors.

Now,

$$\begin{aligned}
\lambda_k(A) &= \min_{\dim(U)=n-k+1} \max_{0\neq x\in U} \frac{x^TAx}{\|x\|_2} \\
&= \min_{\dim(U)=n-k+1} \max_{0\neq x\in U} \frac{x^T(A-B+B)x}{\|x\|_2} \\
&= \min_{\dim(U)=n-k+1} \max_{0\neq x\in U} \left[\frac{x^TBx}{\|x\|_2} + \frac{x^T(A-B)x}{\|x\|_2}\right] \\
&\leq \min_{\dim(U)=n-k+1} \max_{0\neq x\in U} \left[\frac{x^TBx}{\|x\|_2} + \|A-B\|\right] \\
&= \lambda_k(B) + \|A-B\|
\end{aligned}$$

So $\lambda_k(A) - \lambda_K(B) \leq \|A-B\|$. *Analogously we have* $\lambda_k(B) - \lambda_K(A) \leq \|A-B\|$

♦

4 What change does a determinant undergo if to each column (beginning with the second) we add the preceding column and at the same time we add the last to the first?

Solution. *Let us write* $a_1, \ldots, a_n$ *for the column vectors of the original matrix* A. *Then the new matrix is* $B = (a_1 + a_2|a_2 + a_3|\cdots|a_n + a_1)$ *and by linearity we have*

$$\det(B) = \sum_{i_1,\ldots,i_n\in\{0,1\}} \det(a_{1+i_1}|\cdots|a_{n+i_n})$$

where addition in indices is done modulo n. *If* $i_1 = \cdots = i_n = 0$ *we have* $\det(a_{1+i_1}|\cdots|a_{n+i_n}) = \det(A)$. *If* $i_1 = \cdots = i_n = 1$ *we have* $\det(a_{1+i_1}|\cdots|a_{n+i_n}) = \det(a_2|a_3|\cdots|a_n|a_1) = (-1)^{n-1}$. *Finally if* $i_s = 0$ *but* $i_t = 1$ *for some* s, t *then there is an* r *with* $i_r = 1, i_{r+1} = 0$. *But then* $(a_{1+i_1}|\cdots|a_{n+i_n})$ *has two equal columns and so its determinant is equal to* 0. *Therefore* $\det(B) = (1 + (-1)^{n-1})\det(A)$.

♦

5 Let $V_1, V_2, W_1, W_2, U_1, U_2 \in \mathbb{K}$-Vect, $V_1 \xrightarrow{\alpha_1} W_1 \xrightarrow{\beta_1} U_1$, $V_2 \xrightarrow{\alpha_2} W_2 \xrightarrow{\beta_2} U_2$ $\mathbb{K}$-linear. Prove that

$$(\beta_1 \otimes \beta_2)(\alpha_1 \otimes \alpha_2) = (\beta_1\alpha_1) \otimes (\beta_2\alpha_2)$$

Solution. *Recall the general definition of the tensor product of linear maps, we have successively:*

$$\begin{aligned}
((\beta_1\alpha_1)\otimes(\beta_2\alpha_2))(v_1\otimes v_2) &= (\beta_1\alpha_1)(v_1)\otimes(\beta_2\alpha_2)(v_2) \\
&= \beta_1(\alpha_1(v_1))\otimes\beta_2(\alpha_2(v_2)) \\
&= (\beta_1\otimes\beta_2)(\alpha_1(v_1)\otimes\alpha_2(v_2)) \\
&= ((\beta_1\otimes\beta_2)(\alpha_1\otimes\alpha_2))(v_1\otimes v_2)
\end{aligned}$$

Thus, the two linear maps $V_1 \otimes V_2 \to U_1 \otimes U_2$ *are equal when composed with the canonical bilinear map* $V_1 \times V_2 \to V_1 \otimes V_2$*, hence equal (by the universal property).*

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3716`*.*

PART 2

Calculus

1 Let $a \in \mathbb{R}$. Prove that

$$\sum_{n=1}^{\infty} 2^{2n} \sin^4 \frac{a}{2^n} = a^2 - \sin^2 a$$

Solution. *First of all we note that*

$$\begin{aligned}
2^{2n} \sin^4 \frac{a}{2^n} &= 2^{2n} \cdot \sin^2 \frac{a}{2^n} \cdot \sin^2 \frac{a}{2^n} \\
&= 2^{2n} \cdot \sin^2 \frac{a}{2^n} \cdot \left(1 - \cos^2 \frac{a}{2^n}\right) \\
&= 2^{2n} \cdot \sin^2 \frac{a}{2^n} - 2^{2n} \cdot \sin^2 \frac{a}{2^n} \cos^2 \frac{a}{2^n} \\
&= 2^{2n} \cdot \sin^2 \frac{a}{2^n} - 2^{2n-2} \sin^2 \frac{a}{2^{n-1}}
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{n=1}^{m} 2^{2n} \sin^4 \frac{a}{2^n} &= \sum_{n=1}^{m} \left(2^{2n} \cdot \sin^2 \frac{a}{2^n} - 2^{2n-2} \sin^2 \frac{a}{2^{n-1}}\right) \\
&= 2^{2m} \sin^2 \frac{a}{2^m} - \sin^2 a
\end{aligned}$$

Letting $m \to +\infty$ we get the requested value.

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3528`.

2 Let $\xi \in \mathbb{R}$ and $a \in (0, 1)$. Show that

$$\int_{-\infty}^{\infty} e^{-2\pi i x \xi} \frac{\sin \pi a}{\cosh \pi x + \cos \pi a} \, dx = \frac{2 \sinh 2\pi a \xi}{\sinh 2\pi \xi}$$

Solution. *We note that*

$$\begin{aligned}
\frac{\sin \pi a}{\cosh \pi x + \cos \pi a} &= \frac{1}{1 + e^{-\pi(x+ia)}} - \frac{1}{1 + e^{-\pi(x-ia)}} \\
&= 2 \sum_{n=1}^{\infty} (-1)^n e^{-n\pi x} \sin n\pi a
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{e^{-2\pi i \xi x} \sin \pi a}{\cosh \pi x + \cos \pi a} \, dx &= \int_{0}^{\infty} \frac{2 \sin \pi a \cos 2\pi \xi x}{\cosh \pi x + \cos \pi a} \, dx \\
&= 4 \int_{0}^{\infty} \sum_{n=1}^{\infty} (-1)^n e^{-n\pi x} \cos 2\pi \xi x \sin n\pi a \, dx \\
&= \frac{4}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{n \sin n\pi a}{n^2 + 4\xi^2}
\end{aligned}$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n+2i\xi}+\frac{1}{n-2i\xi}\right)\sin n\pi a$$
$$= \frac{2}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n \sin n\pi a}{n+2i\xi}$$
$$= \frac{2\sin 2\pi i a\xi}{\sin 2\pi i\xi} \qquad \textit{(by residue theorem)}$$
$$= \frac{2\sinh 2\pi a\xi}{\sinh 2\pi\xi}$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3597`.

3 Prove that

$$\sum_{n=1}^{\infty}\frac{1}{\sinh^2 n\pi} = \frac{1}{6}-\frac{1}{2\pi}$$

Solution. *Let us consider the function*

$$f(z) = \frac{\cot \pi z}{\sinh^2 \pi z}$$

and integrate it along a quadratic counterclockwise contour Γ_N *with vertices* $\frac{N}{2}(\pm 1 \pm i)$ *where* N *is a big odd natural number. Hence,*

$$\oint_{\Gamma_N} f(z)\,\mathrm{d}z = \int_{N/2}^{-N/2} f\left(x+\frac{iN}{2}\right)\mathrm{d}x + i\int_{N/2}^{-N/2} f\left(iy+\frac{N}{2}\right)\mathrm{d}y$$
$$+ \int_{-N/2}^{N/2} f\left(x-\frac{iN}{2}\right)\mathrm{d}x + i\int_{-N/2}^{N/2} f\left(iy+\frac{N}{2}\right)\mathrm{d}y$$

We note that $\lim\limits_{N\to+\infty} f\left(x\pm\frac{iN}{2}\right) = \frac{\mp i}{\cosh^2 \pi x}$ *and that* $\lim\limits_{N\to+\infty} f\left(iy\pm\frac{N}{2}\right) = 0$. *It's also easy to see that*

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{\cosh^2 \pi x} = \left[\frac{\tanh \pi x}{\pi}\right]_{-\infty}^{\infty} = \frac{2}{\pi}$$

Hence, as $N \to +\infty$ *we have that*

$$\oint_{\Gamma_N} f(z)\,\mathrm{d}z = -\frac{4i}{\pi}$$

By Residue theorem we have that

$$\oint_{\Gamma_N} f(z)\,\mathrm{d}z = 2\pi i\sum \mathfrak{Res}\,(f; z=k) + \mathfrak{Res}\,(f; z=ik)$$

It is straightforward to show that

$$\mathfrak{Res}\,(f; z = k) = \mathfrak{Res}\,(f; z = ik) = \frac{1}{\pi \sinh^2 \pi k}$$
$$\mathfrak{Res}\,(f; z = 0) = -\frac{2}{3\pi}$$

Hence,

$$\oint_{\Gamma_N} f(z)\,\mathrm{d}z = 8i \sum_{k=1}^{\infty} \frac{1}{\sinh^2 n\pi} - \frac{4i}{3}$$

in the limit $N \rightarrow +\infty$. The result follows.

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3604`.

(4) Let $\psi^{(0)}$ denote the digamma series. Evaluate the series

$$\mathcal{S} = \sum_{n=1}^{\infty} \left(\psi^{(0)} \left(\frac{1+n}{2} \right) - \psi^{(0)} \left(\frac{n}{2} \right) - \frac{1}{n} \right)$$

Solution. *We have successively:*

$$\begin{aligned}
\mathcal{S} &= \sum_{n=1}^{\infty} \left(\psi^{(0)} \left(\frac{1+n}{2} \right) - \psi^{(0)} \left(\frac{n}{2} \right) - \frac{1}{n} \right) \\
&= \lim_{N \rightarrow +\infty} \left(\psi^{(0)} \left(\frac{N+1}{2} \right) - \psi^{(0)} \left(\frac{1}{2} \right) - \mathcal{H}_N \right) \\
&= \lim_{N \rightarrow +\infty} \left(\ln \frac{N}{2} + 2\ln 2 + \gamma - \ln N - \gamma + \mathcal{O}\left(\frac{1}{N} \right) \right) \\
&= \ln 2
\end{aligned}$$

since $\psi^{(0)}\left(\frac{1}{2}\right) = -2\ln 2 - \gamma$, $\mathcal{H}_n = \gamma + \ln n + \mathcal{O}\left(\frac{1}{n}\right)$ *and* $\psi^{(0)}(x) = \ln x + \mathcal{O}\left(\frac{1}{x}\right)$.

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3648`.

(5) Let ζ denote the Riemann zeta function. Evaluate the integral

$$\mathcal{J} = \int_0^1 \int_0^1 \cdots \int_0^1 \ln \prod_{k=1}^{n} x_k \ln \left(1 - \prod_{k=1}^{n} x_k \right) \mathrm{d}\left(x_1, x_2, \ldots, x_n\right)$$

Solution. *Based on symmetries,*

$$\begin{aligned}
\mathcal{J} &= n \int_0^1 \int_0^1 \cdots \int_0^1 \ln x_1 \ln \left(1 - \prod_{k=1}^{n} x_k \right) \mathrm{d}\left(x_1, x_2, \ldots, x_n\right) \\
&= -n \int_0^1 \int_0^1 \cdots \int_0^1 \ln x_1 \sum_{i=1}^{\infty} \frac{(x_1 \ldots x_n)^i}{i} \,\mathrm{d}\left(x_1, x_2, \ldots, x_n\right) \\
&= -n \sum_{i=1}^{\infty} \frac{1}{i(i+1)^{n-1}} \int_0^1 x_1^i \ln x_1 \, dx_1
\end{aligned}$$

$$= n \sum_{i=1}^{\infty} \frac{1}{i(i+1)^{n+1}}$$

Let $\mathcal{S}_n = \sum_{i=1}^{\infty} \frac{1}{i(i+1)^{n+1}}$. *It follows that*

$$\begin{aligned}
\mathcal{S}_n &= \sum_{i=1}^{\infty} \frac{1}{(i+1)^n} \left(\frac{1}{i} - \frac{1}{i+1} \right) \\
&= \mathcal{S}_{n-1} - \sum_{i=2}^{\infty} \frac{1}{i^{n+1}} \\
&= \mathcal{S}_{n-1} + 1 - \zeta(n+1)
\end{aligned}$$

Using the recursion we get that

$$\mathcal{S}_n = n + 1 - \sum_{k=2}^{n+1} \zeta(k)$$

Thus,

$$\mathcal{J} = n \left(n + 1 - \sum_{k=2}^{n+1} \zeta(k) \right)$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3666`.

6 Let $\{\cdot\}$ denote the fractional part. Prove that

$$\int_0^{\pi/2} \sin 2x \{\ln^{2n-1} \tan x\} \, dx$$

for the different values of the integer number n.

Solution. *Let* $\mathcal{J}$ *denote the integral,*

$$\begin{aligned}
\mathcal{J} &= \int_0^{\pi/2} \sin 2x \left\{ \ln^{2n-1} \tan x \right\} \, dx \\
&\overset{u=\pi/2-x}{=\!=\!=} \int_0^{\pi/2} \sin 2\left(\frac{\pi}{2} - u \right) \left\{ \ln^{2n-1} \tan \left(\frac{\pi}{2} - u \right) \right\} \, du \\
&= \int_0^{\pi/2} \sin 2u \left\{ \ln^{2n-1} \frac{1}{\tan u} \right\} \, du \\
&= \int_0^{\pi/2} \sin 2u \left\{ - \ln^{2n-1} \tan u \right\} \, du \\
&= \frac{1}{2} \int_0^{\pi/2} \sin 2u \left(\left\{ \ln^{2n-1} \tan u \right\} + \left\{ - \ln^{2n-1} \tan u \right\} \right) \, du \\
&= \frac{1}{2} \int_0^{\pi/2} \sin 2u \, du \\
&= \frac{1}{2}
\end{aligned}$$

since $\{x\}+\{-x\}=1$ *if* $x\notin\mathbb{Z}$ *whereas* $\{x\}+\{-x\}=0$ *if* $x\in\mathbb{Z}$. *Therefore,*

$$\left\{\ln^{2n-1}\tan u\right\}+\left\{-\ln^{2n-1}\tan u\right\}=1$$

except of a countable set $A=\left\{x_k\in\left(0,\frac{\pi}{2}\right)\middle|\ln^{2n-1}\tan x_k\in\mathbb{Z}\right\}$ *whose measure is* 0.

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3672`.

7 Prove that

$$\sum_{a_1=0}^{b_1}\sum_{a_2=0}^{b_2}\cdots\sum_{a_n=0}^{b_n}\frac{\binom{b_1}{a_1}\binom{b_2}{a_2}\cdots\binom{b_n}{a_n}}{\binom{b_1+b_2+\ldots+b_n}{a_1+a_2+\ldots+a_n}}=b_1+b_2+\cdots+b_n+1$$

Solution. *We may begin with the beta function identity for non negative integer values of* a,b.

$$\int_0^1 x^{b-a}(1-x)^a\,\mathrm{d}x=\frac{1}{(b+1)\binom{b}{a}}$$

Hence, for non-negative integers a',b'

$$\begin{aligned}\sum_{a=0}^{b}\frac{\binom{b}{a}}{\binom{b+b'}{a+a'}}&=(b+b'+1)\sum_{a=0}^{b}\binom{b}{a}\int_0^1 x^{b+b'-a-a'}(1-x)^{a+a'}\,\mathrm{d}x\\&=(b+b'+1)\int_0^1 x^{b'-a'}(1-x)^{a'}\,\mathrm{d}x\\&=\frac{b+b'+1}{(b'+1)\binom{b'}{a'}}\end{aligned}$$

As a result we may compute the nested summation as,

$$\begin{aligned}\sum_{a_1=0}^{b_1}\cdots\sum_{a_n=0}^{b_n}\frac{\binom{b_1}{a_1}\cdots\binom{b_n}{a_n}}{\binom{b_1+\cdots+b_n}{a_1+\cdots+a_n}}&=\frac{b_1+\cdots+b_n+1}{b_1+\cdots+b_{n-1}+1}\sum_{a_1=0}^{b_1}\cdots\sum_{a_{n-1}=0}^{b_{n-1}}\frac{\binom{b_1}{a_1}\cdots\binom{b_{n-1}}{a_{n-1}}}{\binom{b_1+\cdots+b_{n-1}}{a_1+\cdots+a_{n-1}}}\\&=\cdots\\&=\prod_{j=0}^{n-2}\frac{b_1+\cdots+b_{n-j}+1}{b_1+\cdots+b_{n-j-1}+1}\sum_{a_1=0}^{b_1}\frac{\binom{b_1}{a_1}}{\binom{b_1}{a_1}}\\&=b_1+\cdots+b_n+1\end{aligned}$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3688`.

8 Let $\{F_n\}_{n\in\mathbb{N}}$ denote the Fibonacci sequence such that $F_1=F_2=1$ and $F_3=2$. Prove that

$$\sum_{n=2}^{\infty}\operatorname{arctanh}\frac{1}{F_{2n}}=\frac{\log 3}{2}$$

Solution. *It holds that*

$$\operatorname{arctanh} x - \operatorname{arctanh} y = \operatorname{arctanh}\left(\frac{x-y}{1-xy}\right)$$

It follows from Cassini's identity that

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n \tag{1}$$

Setting $n \mapsto 2n$ *back at* (1) *we get*

$$F_{2n}^2 = F_{2n-1}F_{2n+1} - 1 \tag{2}$$

Since $F_n = F_{n-1} + F_{n-2}$ *we get*

$$F_{2n+1} = F_{2n} + F_{2n-1} \Leftrightarrow F_{2n} = F_{2n+1} - F_{2n-1} \tag{3}$$

Hence,

$$\begin{aligned}
\sum_{n=2}^{\infty} \operatorname{arctanh} \frac{1}{F_{2n}} &= \sum_{n=2}^{\infty} \operatorname{arctanh} \frac{F_{2n}}{F_{2n}^2} \\
&= \sum_{n=2}^{\infty} \operatorname{arctanh} \frac{F_{2n-1} - F_{2n+1}}{1 - F_{2n-1}F_{2n+1}} \\
&= \sum_{n=2}^{\infty} \operatorname{arctanh} \frac{\frac{1}{F_{2n-1}} - \frac{1}{F_{2n+1}}}{1 - \frac{1}{F_{2n-1}F_{2n+1}}} \\
&= \lim_{m\to+\infty} \sum_{n=2}^{m} \left[\operatorname{arctanh} \frac{1}{F_{2n-1}} - \right. \\
&\qquad \left. - \operatorname{arctanh} \frac{1}{F_{2n+1}}\right] \\
&= \lim_{m\to+\infty} \left[\operatorname{arctanh} \frac{1}{F_3} - \operatorname{arctanh} \frac{1}{F_{2m+1}}\right] \\
&= \operatorname{arctanh} \frac{1}{2} \\
&= \frac{\log 3}{2}
\end{aligned}$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3705`.

9 Evaluate

$$\Omega = \prod_{n=1}^{\infty} \frac{1}{1 - \tan^2 2^{-n}}$$

Solution. *Let*

$$a_n = \frac{1}{1 - \tan^2(2^{-n})} = \frac{\cos^2(2^{-n})}{\cos^2(2^{-n}) - \sin^2(2^{-n})} = \frac{\cos^2(2^{-n})}{\cos(2^{-n+1})}$$

Hence,

$$\begin{aligned}\prod_{i=1}^{n} a_i &= \frac{\cos^2\left(2^{-1}\right)}{\cos\left(2^{0}\right)} \cdot \frac{\cos^2\left(2^{-2}\right)}{\cos\left(2^{-1}\right)} \cdots \frac{\cos^2\left(2^{-n}\right)}{\cos\left(2^{-n+1}\right)} \\ &= \frac{1}{\cos 1} \cos\left(\frac{1}{2}\right)\cos\left(\frac{1}{4}\right)\cdots\cos\left(\frac{1}{2^n}\right) \\ &= \frac{1}{\cos 1} \frac{\cos\left(\frac{1}{2}\right)\cos\left(\frac{1}{4}\right)\cdots 2\sin\left(\frac{1}{2^n}\right)\cos\left(\frac{1}{2^n}\right)}{2\sin\left(\frac{1}{2^n}\right)} \\ &= \frac{1}{\cos 1}\cos 1 \frac{\cos\left(\frac{1}{2}\right)\cos\left(\frac{1}{4}\right)\cdots\cos\left(\frac{1}{2^{n-1}}\right)\sin\left(\frac{1}{2^{n-1}}\right)}{2\sin\left(\frac{1}{2^n}\right)} \\ &= \frac{1}{\cos 1}\frac{\sin 1}{2^n\sin\left(\frac{1}{2^n}\right)} \\ &= \frac{\tan 1}{2^n\sin\left(\frac{1}{2^n}\right)}\end{aligned}$$

Hence,

$$\Omega = \lim_{n\to+\infty}\prod_{i=1}^{n} a_i = \tan 1$$

♦

(10) Let $\mathbf{Li}_n$ denote the polylogarithm. Prove that

$$\sum_{n=2}^{\infty}\left(\mathrm{Li}_n(1) - 1\right) = 1$$

Solution. *Since* $\zeta(n) = \sum_{k=1}^{\infty}\frac{1}{k^n}$ *we have successively*

$$\begin{aligned}\sum_{n=2}^{\infty}\left(\mathrm{Li}_n(1)-1\right) &= \sum_{n=2}^{\infty}\left(\zeta(n)-1\right) \\ &= \sum_{n=2}^{\infty}\sum_{k=2}^{\infty}\frac{1}{k^n} \\ &= \sum_{k=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{k^n} \\ &= \sum_{k=2}^{\infty}\left(\frac{1}{k-1}-\frac{1}{k}\right) \\ &= 1\end{aligned}$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3764`.

(11) Evaluate the integral

$$\mathcal{J} = \int_0^1 \int_0^1 (xy)^{xy}\, \mathrm{d}(x, y)$$

Solution. *Let* $t = xy$ *and* $s = y$. *The Jacobian is*

$$\frac{\partial(s, t)}{\partial(x, y)} = y \Rightarrow \left(\frac{\partial(s, t)}{\partial(x, y)}\right)^{-1} = \frac{1}{y} = \frac{1}{s}$$

Hence,

$$\begin{aligned}
\mathcal{J} &= \iint\limits_{[0,1]^2} (xy)^{xy}\, \mathrm{d}(x, y) \\
&= \int_0^1 \int_0^s \frac{t^t}{s}\, \mathrm{d}(t, s) \\
&= \int_0^1 \int_t^1 \frac{t^t}{s}\, \mathrm{d}(s, t) \\
&= -\int_0^1 t^t \log t \, \mathrm{d}t
\end{aligned}$$

However , since $\displaystyle\int_0^1 t^t (1 + \log t)\, \mathrm{d}t = 0$ *we conclude that*

$$\mathcal{J} = \int_0^1 t^t \, \mathrm{d}t = \mathcal{S}$$

where $\mathcal{S}$ *is Sophomore's dream constant.*

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3810`.

PART 3

Analysis

1 Let $\{a_n\}_{n\in\mathbb{N}}$ be a real sequence such that

> If $\{b_n\}_{n\in\mathbb{N}}$ is a real sequence that is square summable; i.e $\sum\limits_{n=1}^{\infty} b_n^2 < +\infty$ the sequence $\sum\limits_{n=1}^{\infty} a_n b_n$ converges.

Prove that $\{a_n\}_{n\in\mathbb{N}}$ is also square summable.

Solution. *Let $f_N : \ell_2 \to \mathbb{R}$ be defined as*

$$f_N(b) = \sum_{n=1}^{N} a_n b_n$$

where $\mathbf{b} = (b_n) \in \ell_2$. We note that

$$\begin{aligned} |f_N(b)|^2 &\leq \sum_{n=1}^{N} |a_n|^2 \sum_{n=1}^{N} |b_n|^2 \\ &\leq \|b\|^2 \sum_{n=1}^{N} |a_n|^2 \end{aligned}$$

Equality holds when $\mathbf{b} = (a_1, \ldots, a_N, 0, 0, \ldots)$. Hence, $\|f_N\| = \left(\sum\limits_{n=1}^{N} |a_n|^2\right)^{1/2}$. From the hypothesis, it follows that $\{f_n\}_{n\in\mathbb{N}}$ is pointwise bounded. It follows from the Uniform boundedness principle (Banach - Steinhaus) that $\|f_N\| = \left(\sum\limits_{n=1}^{N} |a_n|^2\right)^{1/2}$ are bounded. Hence, $\{a_n\}_{n\in\mathbb{N}}$ is square summable.

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3581`.

2 Let gpf denote the greatest prime factor of n. For example $\text{gpf}(17) = 17$, $\text{gpf}(18) = 3$. Define $\text{gpf}(1) = 1$. Examine if the sum

$$\mathcal{S} = \sum_{n=1}^{\infty} \frac{1}{n\, \text{gpf}(n)}$$

converges.

Solution. *Let p_k be the k-th prime number.*

$$\sum_{n\geq 1} \frac{1}{n\, \text{gpf}(n)} = \sum_{k\geq 1} \frac{1}{p_k} \sum_{\text{gpf}(n)=p_k} \frac{1}{n}$$

If $\text{gpf}(n) = p_k$, then $n = p_1^{i_1} \ldots p_{k-1}^{i_{k-1}} p_k^{i_k}$ with $i_j \geq 0$, $1 \leq j < k$ and $i_k \geq 1$. It folows that

$$\sum_{\text{gpf}(n)=p_k} \frac{1}{n} = \left(\sum_{i_1\geq 0} p_1^{-i_1}\right) \cdots \left(\sum_{i_{k-1}\geq 0} p_{k-1}^{-i_{k-1}}\right)\left(\sum_{i_k\geq 1} p_k^{-i_k}\right)$$

$$= \frac{1}{p_k} \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right)^{-1}$$

From Merten's theorem

$$\prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right)^{-1} \sim e^{\gamma} \log p_k$$

and the original series has the same character as

$$\sum_{k=1}^{\infty} \frac{\log p_k}{p_k^2}$$

which is convergent.

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3587`.

3 Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x + \pi) = f(x - \pi) = f(x)$ for all $x \in \mathbb{R}$. Prove that

$$\int_{-\infty}^{\infty} f(x) \frac{\sin^2 x}{x^2} \, \mathrm{d}x = \int_0^{\pi} f(x) \, \mathrm{d}x$$

Solution. *We have successively:*

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x) \frac{\sin^2 x}{x^2} \, \mathrm{d}x &= \sum_{k=-\infty}^{\infty} \int_{k\pi}^{(k+1)\pi} f(x) \frac{\sin^2 x}{x^2} \, \mathrm{d}x \\
&= \sum_{k=-\infty}^{\infty} \int_0^{\pi} f\,(t + k\pi) \, \frac{\sin^2 (t + k\pi)}{(t + k\pi)^2} \, \mathrm{d}t \\
&= \sum_{k=-\infty}^{\infty} \int_0^{\pi} f(t) \frac{(-1)^{2k} \sin^2 t}{(t + k\pi)^2} \, \mathrm{d}t \\
&= \int_0^{\pi} f(t) \sin^2 t \sum_{k=-\infty}^{\infty} \frac{1}{(t + k\pi)^2} \, \mathrm{d}t \\
&= \int_0^{\pi} f(t) \, \mathrm{d}t
\end{aligned}$$

since

$$\sum_{k=-\infty}^{\infty} \frac{1}{(t + k\pi)^2} = \csc^2 t$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3607`.

4 If f is an even continuous function defined on $[1, 1]$ and all its midpoint Riemann sums are zero (i.e $\sum_{k=1}^{n} f\left(\frac{2k-1}{n} - 1\right) \cdot \frac{2}{n} = 0$ for every $n \in \mathbb{N}$), then is $f \equiv 0$?

Solution. *Let* $a_n : \mathbb{N} \cup \{0\} \to \mathbb{R}$ *be an absolutely summable sequence, and use it to define a function* $f : [-1,1] \to \mathbb{R}$ *as*

$$f(x) = \sum_{n=0}^{\infty} a_n \cos n\pi x$$

It is evident that f *is even. By the Weierstrass M-Test* $\sum_{n=0}^{\infty} a_n \cos n\pi x$ *converges uniformly on* $[-1,1]$ *to* f*. Hence,* f *is continuous, and using the fact that* $\int_{-1}^{1} \cos n\pi x \,\mathrm{d}x = 0$ *we have that*

$$\begin{aligned} \int_{-1}^{1} f(x)\,\mathrm{d}x &= \int_{-1}^{1} \sum_{n=0}^{\infty} a_n \cos n\pi x \,\mathrm{d}x \\ &= \sum_{n=0}^{\infty} \int_{-1}^{1} a_n \cos n\pi x \,\mathrm{d}x \\ &= 2a_0 \end{aligned}$$

Next, we make two observations:

- *If* $\sum_{k=1}^{N} f\left(\frac{2k-1}{N} - 1\right) \cdot \frac{2}{N} = 0$ *for every* $N \in \mathbb{N}$ *then by the definition of the Riemann integral we have that*

$$\begin{aligned} 2a_0 &= \int_{-1}^{1} f(x)\,\mathrm{d}x \\ &= \lim_{N\to+\infty} \sum_{k=1}^{N} f\left(\frac{2k-1}{N} - 1\right) \cdot \frac{2}{N} \\ &= 0 \end{aligned}$$

  which yields $a_0 = 0$.
- *Using the identity*

$$\sum_{k=1}^{N} \cos\left(n\pi\left(\frac{2k-1}{N} - 1\right)\right) = \begin{cases} (-1)^{n(N+1)/N} N & , \quad N \mid n \\ 0 & , \quad N \nmid n \end{cases}$$

  we obtain the following string of bi-implications for every $N \in \mathbb{N}$

$$\begin{aligned} \sum_{k=1}^{N} f\left(\frac{2k-1}{N} - 1\right) \cdot \frac{2}{N} = 0 &\Leftrightarrow \sum_{k=1}^{N} \left(\sum_{n=0}^{\infty} a_n \cos\left(n\pi\left(\frac{2k-1}{N} - 1\right)\right) \frac{2}{N} \right) = 0 \\ &\Leftrightarrow \sum_{n=0}^{\infty} \left(\sum_{k=1}^{N} a_n \cos\left(n\pi\left(\frac{2k-1}{N} - 1\right)\right) \frac{2}{N} \right) = 0 \\ &\Leftrightarrow \sum_{n=0}^{\infty} a_n \sum_{k=1}^{N} \cos\left(n\pi\left(\frac{2k-1}{N} - 1\right)\right) \frac{2}{N} = 0 \\ &\Leftrightarrow \sum_{\substack{n\in\mathbb{N}\cup\{0\} \\ N\mid n}} a_n (-1)^{n(N+1)/N} N \cdot \frac{2}{N} = 0 \\ &\Leftrightarrow \sum_{\substack{n\in\mathbb{N}\cup\{0\} \\ N\mid n}} a_n (-1)^{n(N+1)/N} = 0 \end{aligned}$$

In order for all of the mid-point Riemann sums of f to be zero, it is thus necessary and sufficient that

(i) $a_0 = 0$ *and*

(ii) $\displaystyle\sum_{\substack{n\in\mathbb{N}\cup\{0\} \\ N|n}} a_n(-1)^{n(N+1)/N} = 0$

For this reason we choose the sequence a_n as follows

$$a_n = \begin{cases} 0 & , \quad \textit{if there exists no } j \in \mathbb{N}\cup\{0\} \textit{ such that } n = 2^j \\ -1 & , \quad n = 1 \\ \dfrac{1}{n} & , \quad \textit{if there exists a } j \in \mathbb{N} \textit{ such that } n = 2^j \end{cases}$$

Finally, f is continuous , even , non zero and its Riemann mid-points are zero. To see that f is non zero we simply calculate $f(1)$;

$$f(1) = -\cos\pi + \sum_{j=1}^{\infty}\frac{\cos 2^j\pi}{2^j} = 1 + \sum_{j=1}^{\infty}\frac{1}{2^j} = 2$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3614`.

5 Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous and strictly increasing function. Consider the line $y = ax+\beta$, $a > 0$. Prove that:

(i) $\lim\limits_{x\to+\infty}(f(x) - ax - \beta) = -\infty$.

(ii) C_f has a unique intersection point with the line above.

Solution. *(i) As $x \to +\infty$ we can without loss of generality assume that $x \geq 0$, hence $f(x) \leq f(0)$. Thus,*

$$f(x) - ax - \beta \leq f(0) - ax - \beta = -ax - \gamma \to -\infty$$

Similarly , we can prove that $\lim\limits_{x\to-\infty}(f(x) - ax - \beta) = +\infty$.

(ii) The function $h(x) = f(x) - ax - \beta$, $x \in \mathbb{R}$ is continuous and strictly decreasing. It follows from question (i.) that $h(\mathbb{R}) = \mathbb{R}$. Since $0 \in \mathbb{R}$ it follows that there exists an $x_0 \in \mathbb{R}$ such that $h(x_0) = 0$ which is unique due to the monotony of h.

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3622`.

6 Let $f:\mathbb{R}^n\to\mathbb{R}$ be differentiable on $\mathbb{R}^n \setminus \{0\}$ and continuous at 0 . If

$$\lim_{x\to 0}\frac{\partial f}{\partial x_i}(x) = 0$$

for $i = 1, 2, \ldots, n$ then prove that f is differentiable at 0.

Solution. *Let us assume without loss of generality that* $f(0) = 0$. *We will show that* $f(x) = o(\|x\|_\infty)$ *as* $x \to 0$ *and that means that* f *is differentiable at* $x = 0$ *with* $f'(0) = 0$.

Fix $\epsilon > 0$ *and choose* $\delta \in \left(0, \frac{1}{n}\right)$ *such that* $0 < \|x\|_\infty < \delta$, $1 \leq i \leq n$. *Therefore* $\left|f_{x_i}\right| < \epsilon$. *Now suppose* $\|x\|_\infty < \delta$. *Let* a_k , $0 \leq k \leq n$ *be the point that coincides with* x *on the first* k *coordinates and is* 0 *elsewhere. Then* $a_0, \ldots, a_n$ *is a path from* 0 *to* x *and each vector* $a_{k+1} - a_k$ *is parallel to one of the axes. Hence*

$$\begin{aligned} |f(x)| &= \left|\sum_{k=0}^{n-1} (f(a_k) - f(a_{k-1})\right| \\ &\leq \sum_{k=0}^{n-1} |f(a_k) - f(a_{k-1})| \\ &\leq n\epsilon \, \|a_k - a_{k-1}\|| \\ &< n\delta\epsilon \\ &< \epsilon \end{aligned}$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3636`.

7 Let f be a continuous real-valued function on $\mathbb{R}$ satisfying

$$|f(x)| \leq \frac{1}{1+x^2} \qquad \forall x$$

Define a function F on $\mathbb{R}$ by

$$F(x) = \sum_{n=-\infty}^{\infty} f(x+n)$$

(i) Prove that F is continuous and periodic with period 1.

(ii) Prove that if G is continuous and periodic with period 1 then

$$\int_0^1 F(x)G(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} f(x)G(x)\,\mathrm{d}x$$

Solution. *(i) We note that*

$$g(x+1) = \sum_{n=-\infty}^{\infty} f(n+x+1) = \sum_{n'=-\infty}^{\infty} f(x+n) = g(x)$$

(ii) First of all G *is bounded on* $[0, 1]$ *and* $\sum_{n=-N}^{N} f(x+n) \to F(x)$ *uniformly. Hence,*

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)G(x)\,\mathrm{d}x &= \sum_{n=-\infty}^{\infty} \int_n^{n+1} f(x)G(x)\,\mathrm{d}x \\ &= \sum_{n=-\infty}^{\infty} \int_0^1 f(x+n)G(x)\,\mathrm{d}x \end{aligned}$$

$$= \int_0^1 F(x)G(x)\,\mathrm{d}x$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3639`.

(8) Let $\psi^{(1)}$ denote the trigamma function and let f be integrable on $(0,1)$. It holds that

$$\int_0^1 f\left(\left\{\frac{1}{x}\right\}\right)\mathrm{d}x = \int_0^1 f(x)\psi^{(1)}(1+x)\,\mathrm{d}x$$

Solution. *We have successively:*

$$\begin{aligned}
\int_0^1 f\left(\left\{\frac{1}{x}\right\}\right)\mathrm{d}x &= \sum_{k=1}^{\infty}\int_{1/(k+1)}^{1/k} f\left(\left\{\frac{1}{x}\right\}\right)\mathrm{d}x \\
&= \sum_{k=1}^{\infty}\int_k^{k+1} f(\{u\})\,\frac{\mathrm{d}u}{u^2} \\
&= \sum_{k=1}^{\infty}\int_k^{k+1} f(u-k)\,\frac{\mathrm{d}u}{u^2} \\
&= \sum_{k=1}^{\infty}\int_0^1 f(v)\,\frac{\mathrm{d}v}{(v+k)^2} \\
&= \int_0^1 f(v)\sum_{k=1}^{\infty}\frac{1}{(v+k)^2}\,\mathrm{d}v \\
&= \int_0^1 f(v)\psi^{(1)}(v+1)\,\mathrm{d}v
\end{aligned}$$

where the interchange between the infinite sum and the integration is allowed by the uniform bound

$$\left|\sum_{k=1}^{N}\frac{1}{(v+k)^2}\right| < \frac{\pi^2}{6} \quad , \quad N \geq 1,\ 0 \leq v \leq 1$$

The result follows.

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3656`.

(9) Evaluate the integral

$$\mathcal{J}_{n,k} = \oint\limits_{|z|=1} \frac{(1+z)^n}{z^{k+1}}\,\mathrm{d}z$$

Solution. *The function* $\dfrac{(1+z)^n}{z^{k+1}}$ *is meromorphic on* $\mathbb{C}$. *Its only pole is* 0 *of order* $k+1$. *Hence,*

$$\mathfrak{Res}\,(f; z=0) = \frac{1}{k!}\lim_{z\to 0}\left(z^{k+1}f(z)\right)^{(k)}$$

$$
\begin{aligned}
&= \frac{1}{k!} \lim_{z \to 0} \left((1+z)^n\right)^{(k)} \\
&= \frac{n(n-1)\cdots(n-k+1)}{k!} \\
&= \binom{n}{k}
\end{aligned}
$$

Therefore,

$$\mathcal{J}_{n,k} = 2\pi i \binom{n}{k}$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3693`.

10 Let f be a convex function on a convex domain Ω and g a convex non-decreasing function on $\mathbb{R}$. Prove that the composition of $g \circ f$ is convex on Ω.

Solution. *We want to prove that for* $x, y \in \Omega$ *it holds that*

$$(g \circ f)(\lambda x + (1-\lambda)y) \leq \lambda (g \circ f)(x) + (1-\lambda)(g \circ f)(y)$$

We have:

$$
\begin{aligned}
(g \circ f)(\lambda x + (1-\lambda)y) &= g\left(f(\lambda x + (1-\lambda)y)\right) \\
&\leq g\left(\lambda f(x) + (1-\lambda) f(y)\right) && (f \text{ convex and } g \text{ nondecreasing}) \\
&\leq \lambda g(f(x)) + (1-\lambda) g(f(y)) && (g \text{ convex}) \\
&= \lambda (g \circ f)(x) + (1-\lambda)(g \circ f)(y)
\end{aligned}
$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3761`.

11 Let $f : (0, +\infty) \to \mathbb{R}$ be a twice differentiable function such that

$$x^2 f''(x) + x f'(x) = x - 2 \quad \text{for all} \quad x > 0$$

If $f(1) = 0$, $f'(1) = 1$ find an explicit formula of f.

Solution. *We have successively*

$$
\begin{aligned}
x^2 f''(x) + x f'(x) = x - 2 &\Rightarrow x f''(x) + f'(x) = 1 - \frac{2}{x} \\
&\Rightarrow (x f'(x))' = (x - 2\ln x)' \\
&\Rightarrow x f'(x) = x - 2\ln x + c_1 \\
&\xRightarrow{x=1 \Rightarrow c_1 = 0} x f'(x) = x - 2\ln x \\
&\Rightarrow f'(x) = 1 - \frac{2\ln x}{x}
\end{aligned}
$$

$$\begin{aligned}&\Rightarrow (f(x))' = \left(x - \ln^2 x\right)' \\ &\Rightarrow f(x) = x - \ln^2 x + c \\ &\xrightarrow{x=1 \Rightarrow c=-1} f(x) = x - \ln^2 x - 1\end{aligned}$$

♦

Exercise lies in https://www.math.tolaso.com.gr/?p=3766.

(12) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f(0) = 1$ and

$$f'(x)\sqrt{x^2+1} = f(x) \quad \text{for all} \quad x \in \mathbb{R}$$

Find an explicit formula for f.

Solution. *We have successively*

$$\begin{aligned} f'(x)\sqrt{x^2+1} = f(x) &\Leftrightarrow f'(x)\sqrt{x^2+1} - f(x) = 0 \\ &\Leftrightarrow f'(x)\sqrt{x^2+1} - xf'(x) + \frac{xf(x)}{\sqrt{x^2+1}} - f(x) = 0 \\ &\Leftrightarrow f'(x)\left(\sqrt{x^2+1} - x\right) + f(x)\left(\frac{x}{\sqrt{x^2+1}} - 1\right) = 0 \\ &\Rightarrow \left[f(x)\left(\sqrt{x^2+1} - x\right)\right]' = (c)' \\ &\xrightarrow{f(0)=1 \Rightarrow c=1} f(x) = x + \sqrt{x^2+1}, \ x \in \mathbb{R} \end{aligned}$$

which satisfies the given conditions.

♦

Exercise lies in https://www.math.tolaso.com.gr/?p=3783.

4

PART

Inequalities

(1) Let a_n be a sequence of positive real terms. Prove that:

$$\frac{1}{\frac{1}{a_1}}+\frac{2}{\frac{1}{a_1}+\frac{1}{a_2}}+\cdots+\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}<2\sum_{i=1}^{n}a_i$$

Solution. *We are making use of the weighted AM - HM inequality. Hence,*

$$\begin{aligned}
\sum_{k=1}^{n}\frac{k}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots\frac{1}{a_k}} &= \sum_{k=1}^{n}\frac{2}{k+1}\cdot\frac{1+2+\cdots+k}{\frac{1}{a_1}+\frac{2}{2a_2}+\cdots+\frac{k}{ka_k}}\\
&\le \sum_{k=1}^{n}\frac{2}{k+1}\cdot\frac{1\cdot a_1+2\cdot 2a_2+\cdots+k\cdot ka_k}{1+2+\cdots+k}\\
&= \sum_{k=1}^{n}\frac{4}{k\,(k+1)^2}\sum_{i=1}^{k}i^2a_i\\
&= \sum_{i=1}^{n}i^2a_i\sum_{k=i}^{n}\frac{4}{k\,(k+1)^2}\\
&< \sum_{i=1}^{n}i^2a_i\sum_{k=i}^{n}\frac{2(2k+1)}{k^2\,(k+1)^2}\\
&= \sum_{i=1}^{n}i^2a_i\sum_{k=i}^{n}\left(\frac{2}{k^2}-\frac{2}{(k+1)^2}\right)\\
&< \sum_{i=1}^{n}i^2a_i\left(\frac{2}{i^2}-\frac{2}{(n+1)^2}\right)\\
&< \sum_{i=1}^{n}i^2a_i\frac{2}{i^2}\\
&= 2\sum_{i=1}^{n}a_i
\end{aligned}$$

♦

(2) Let $a_1, a_2, a_3, \ldots, a_n$ be positive real numbers whose sum is 1. Prove that

$$\frac{a_1^2}{a_1+a_2}+\frac{a_2^2}{a_2+a_3}+\cdots+\frac{a_n^2}{a_n+a_1}\ge\frac{1}{2}$$

Solution. *By the Cauchy -Schwarz inequality we have:*

$$\begin{aligned}
\frac{a_1^2}{a_1+a_2}+\frac{a_2^2}{a_2+a_3}+\cdots\frac{a_n^2}{a_n+a_1} &= \frac{a_1^2}{(\sqrt{a_1+a_2})^2}+\frac{a_2^2}{(\sqrt{a_2+a_3})^2}+\cdots+\frac{a_n^2}{(\sqrt{a_n+a_1})^2}\\
&\ge \frac{1}{a_1+\cdots+a_n+a_1+\cdots+a_n}\left(\frac{a_1\cdot\sqrt{a_1+a_2}}{\sqrt{a_1+a_2}}+\right.\\
&\quad\left.+\frac{a_2\cdot\sqrt{a_2+a_3}}{\sqrt{a_2+a_3}}+\cdots+\frac{a_n\cdot\sqrt{a_n+a_1}}{\sqrt{a_n+a_1}}\right)\\
&= \frac{a_1+a_2+a_3+\cdots a_n}{2(a_1+a_2+a_3+\cdots a_n)}\\
&= \frac{1}{2}
\end{aligned}$$

♦

3 Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be positive numbers such that

$$a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$$

Prove that

$$\frac{a_1^2}{a_1 + b_1} + \cdots + \frac{a_n^2}{a_n + b_n} \geq \frac{1}{2}(a_1 + \cdots + a_n)$$

Solution. *Using Cauchy-Schwarz inequality*

$$(x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2) \geq (x_1 y_1 + \cdots + x_n y_n)^2$$

for $x_i = \dfrac{a_i}{\sqrt{a_i + b_i}}$ *and* $y_i = \sqrt{a_i + b_i}$ *we obtain*

$$\left(\frac{a_1^2}{a_1 + b_1} + \cdots + \frac{a_n^2}{a_n + b_n}\right)((a_1 + b_1) + \cdots + (a_n + b_n)) \geq (a_1 + \cdots + a_n)^2$$

or equivalently

$$2\left(\frac{a_1^2}{a_1 + b_1} + \cdots + \frac{a_n^2}{a_n + b_n}\right)(a_1 + \cdots + a_n) \geq (a_1 + \cdots + a_n)^2$$

and finally

$$\frac{a_1^2}{a_1 + b_1} + \cdots + \frac{a_n^2}{a_n + b_n} \geq \frac{1}{2}(a_1 + \cdots + a_n)$$

♦

4 Let $x_k, y_k > 0$. Prove that

$$\prod_{k=1}^{n}(x_k + y_k)^{1/n} \geq \prod_{k=1}^{n} x_k^{1/n} + \prod_{k=1}^{n} y_k^{1/n}$$

Solution. *By the inequality of arithmetic and geometric means, we have:*

$$\prod_{k=1}^{n}\left(\frac{x_k}{x_k + y_k}\right)^{1/n} \leq \frac{1}{n}\sum_{k=1}^{n}\frac{x_k}{x_k + y_k}$$

and

$$\prod_{k=1}^{n}\left(\frac{y_k}{x_k + y_k}\right)^{1/n} \leq \frac{1}{n}\sum_{k=1}^{n}\frac{y_k}{x_k + y_k}$$

Hence,

$$\prod_{k=1}^{n}\left(\frac{x_k}{x_k + y_k}\right)^{1/n} + \prod_{k=1}^{n}\left(\frac{y_k}{x_k + y_k}\right)^{1/n} \leq \frac{1}{n}n = 1$$

Clearing denominators then gives the desired result.

♦

5 Let $a, b, c > 0$. Prove that

$$4\sum ab \arctan\frac{c}{b} \le \pi \sum a^2$$

Solution. *Let* $f(x) = \arctan x$. *We easily see that* f *is strictly increasing and concave. Thus, by Jensen's inequality we have that*

$$\begin{aligned}\sum \frac{ab}{ab+bc+ca} \arctan\frac{c}{b} &\le \arctan\left(\sum \frac{ab}{ab+bc+ca}\cdot\frac{c}{b}\right)\\ &= \arctan\left(\sum \frac{ac}{ab+bc+ca}\right)\\ &= \arctan 1\\ &= \frac{\pi}{4}\end{aligned}$$

Further, by the Rearrangement inequality, in particular,

$$4\sum ab\cdot\arctan\frac{c}{b} \le \pi(ab+bc+ca) \le \pi(a^2+b^2+c^2)$$

♦

PART 5

General Mathematics

(1) Let $0 < p \leq q$. Prove that

$$\ln \frac{p}{q} \leq \frac{p-q}{\sqrt{pq}}$$

Solution. *Let* $f(x) = \frac{1}{x}$ *and* $g(x) = 1$. *Thus,*

$$\left(\int_q^p \frac{1}{x}\,\mathrm{d}x\right)^2 \leq \int_q^p \frac{1}{x^2}\,\mathrm{d}x \int_q^p 1\,\mathrm{d}x$$

Thus,

$$\begin{aligned}(\ln p - \ln q)^2 &\leq \left(-\frac{1}{p} + \frac{1}{q}\right)(p-q)\\ &= \frac{(p-q)^2}{pq}\end{aligned}$$

The result follows.

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3697`.

(2) Let m, n be positive numbers with $n > m$. Prove that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{m+n-2k}{n-1} = \binom{n}{m+1}$$

Solution. *Since*

$$\binom{n}{m} = \frac{1}{2\pi i} \oint\limits_{|z|=1} \frac{(z+1)^n}{z^{m+1}}\,\mathrm{d}z$$

we have successively

$$\begin{aligned}\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{m+n-2k}{n-1} &= \frac{1}{2\pi i} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \oint\limits_{|z|=1} \frac{(1+z)^{m+n-2k}}{z^n}\,\mathrm{d}z\\ &= \frac{1}{2\pi i} \oint\limits_{|z|=1} \frac{(z+1)^{m+n}}{z^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\mathrm{d}z}{(z+1)^{2k}}\\ &= \frac{1}{2\pi i} \oint\limits_{|z|=1} \frac{(z+1)^{m+n}}{z^n} \left(1 - \frac{1}{(z+1)^2}\right)^n \mathrm{d}z\\ &= \frac{1}{2\pi i} \oint\limits_{|z|=1} \frac{(z+2)^n}{(z+1)^{n-m}}\,\mathrm{d}z\\ &= \mathfrak{Res}_{z=-1} \frac{(z+2)^n}{(z+1)^{n-m}}\\ &= \lim_{z \to -1} \frac{1}{(n-m-1)!} \frac{\mathrm{d}^{n-m-1}}{\mathrm{d}z^{n-m-1}} \left((z+2)^n\right)\end{aligned}$$

$$= \binom{n}{m+1}$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3699`.

3 Prove that in any triangle $\mathrm{AB\Gamma}$ it holds that

$$\left(\frac{\beta}{\gamma} + \frac{\gamma}{\beta}\right)\cos \mathrm{A} + \left(\frac{\gamma}{a} + \frac{a}{\gamma}\right)\cos \mathrm{B} + \left(\frac{a}{\beta} + \frac{\beta}{a}\right)\cos \Gamma = 3$$

Solution. *Using the law of cosines we have that*

$$\cos A = \frac{\beta^2 + \gamma^2 - a^2}{2\beta\gamma} \tag{1}$$

Hence,

$$\begin{aligned}\left(\frac{\beta}{\gamma} + \frac{\gamma}{\beta}\right)\cos A &= \frac{(\beta^2+\gamma^2)(\beta^2+\gamma^2-a^2)}{2\beta^2\gamma^2} \\ &= \frac{\beta^4+\gamma^4-a^2(\beta^2+\gamma^2)}{2\beta^2\gamma^2} + 1\end{aligned}$$

Let $\mathcal{S}$ *denote the LHS. Then,*

$$\begin{aligned}\mathcal{S} &= 3 + \frac{\beta^4+\gamma^4-a^2(\beta^2+\gamma^2)}{2\beta^2\gamma^2} + \frac{a^4+\beta^4-\gamma^2(\beta^2+a^2)}{2a^2\beta^2} \\ &\quad + \frac{a^4+\gamma^4-\beta^2(a^2+\gamma^2)}{2a^2\gamma^2} \\ &= 3\end{aligned}$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3592`.

4 Prove that every function $f : \mathbb{Q} \to \mathbb{Q}$ can be written as the sum of two $1-1$ functions $g, h : \mathbb{Q} \to \mathbb{Q}$.

Solution. *Let $q_1, q_2, \ldots$ be an enumeration of the rationals. Suppose we have already defined g, h on $q_1, \ldots, q_k$ such that their sum is equal to f on those points and they are 1 to 1 so far. When defining g and h on q_{k+1} it is enough to pick any $q \in \mathbb{Q}$ such that $q \notin \{g(q_1), \ldots, g(q_k)\}$ and $f(q) - q \notin \{h(q_1), \ldots, h(q_k)\}$. We then take $g(q_{k+1}) = q$ and $h(q_{k+1}) = f(q) - q$. This is possible as there are only $2k$ bad choices for q.*

♦

5 Prove that there do not exist four points in $\mathbb{R}^2$ whose pairwise distances are all odd integers.

Solution. *Suppose these four points do exist. We can translate them so that one of them is at the origin. Let the four points be* $0, a, b, c \in \mathbb{R}^2$ *Then:*

$$|a|,\ |b|,\ |c|,\ |d|,\ |a-b|,\ |c-a|,\ |b-c|$$

are all odd integers so their squares are all $1 \pmod 8$*. It follows that:*

$$2a \cdot b = |a|^2 + |b|^2 - |a-b|^2 \equiv 1 \pmod 8$$

Let V *be the* 2×3 *matrix whose columns are* a, b, c*. Consider the Gram Matrix:*

$$B = V^t V = \begin{pmatrix} a\cdot a & a\cdot b & a\cdot c \\ b\cdot a & b\cdot b & b\cdot c \\ c\cdot a & c\cdot b & c\cdot c \end{pmatrix}$$

We have:

$$2B \equiv \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \pmod 8$$

Thus $\det(2B) = 4 \pmod 8$ *and hence* $\det B \neq 0$*. However, this is impossible since*

$$\text{rank} B = \text{rank} V^t V < \text{rank} V < 2$$

as V *is a* 2×3 *matrix.*

♦

(6) Find the geometric mean , with respect to the usual measure on the interval, of all the real numbers in the range $(0, 1]$.

Solution. *The geometric mean with respect to the usual measure on the interval is actually defined as:*

$$\mathcal{GM} = e^{\int_0^1 \ln x\, dx} = e^{-1} = \frac{1}{e}$$

♦

(7) Let Φ denote the golden ratio. Solve the equation

$$x^3 + x^2 - \Phi^5 x + \Phi^5 = 0$$

Solution. *First of all we note that*

$$\begin{cases} \Phi^2 = \Phi + 1 \\ \Phi^3 = \Phi^2 + \Phi = 2\Phi + 1 \end{cases} \Rightarrow \Phi^5 = 2\Phi^2 + 3\Phi + 1 \Rightarrow \boldsymbol{\Phi^5 = 5\Phi + 3}$$

We easily note that Φ *is one root of the equation, hence using Horner we get that*

$$x^2 + \Phi^2 x - (2\Phi^2 + \Phi) = 0 \Leftrightarrow x = \frac{-\Phi^2 \pm \sqrt{9\Phi^2 + 6\Phi + 1}}{2}$$

$$\Leftrightarrow x = \frac{-(\Phi + 1) \pm (3\Phi + 1)}{2}$$

Hence Φ *is a double root and the other root is* $x = -2 - \sqrt{5}$.

♦

8 Let k be a positive integer and g a polynomial of degree at most $k-2$. Show that

$$\sum_{n=0}^{\infty} \frac{g(n)}{(n+1)(n+2)\cdots(n+k)}$$

converges to a rational number.

Solution. *We apply partial fractions decomposition. Hence,*

$$\frac{g(n)}{(n+1)\cdots(n+k)} = \frac{a_1}{n+1} + \cdots + \frac{a_k}{n+k}$$

for rational numbers a_i. *Now, crucially, since* $\deg g \leq k-2$, *we have that* $\sum a_i = 0$. *So we have a telescoping series with only the first few terms surviving, all of them rational.*

♦

9 Let $p, q \in \mathbb{N}$. Find the maximum vale of the function

$$f(x) = \sin^p x \cos^q x$$

Solution. *Using the AM - GM inequality we have*

$$\begin{aligned}
\frac{1}{p+q} &= \frac{\overbrace{\frac{\sin^2 x}{p} + \frac{\sin^2 p}{p} + \cdots + \frac{\sin^2 p}{p}}^{p} + \overbrace{\frac{\cos^2 x}{q} + \frac{\cos^2 x}{q} + \cdots + \frac{\cos^2 x}{q}}^{q}}{p+q} \\
&\geq \sqrt[p+q]{\left(\frac{\sin^2 x}{p}\right)^p \left(\frac{\cos^2 x}{q}\right)^q} \\
&= \sqrt[p+q]{\frac{f^2(x)}{p^p q^q}}
\end{aligned}$$

Hence,

$$f(x) = \sqrt{\frac{p^p q^q}{(p+q)^{p+q}}}$$

Equality holds when

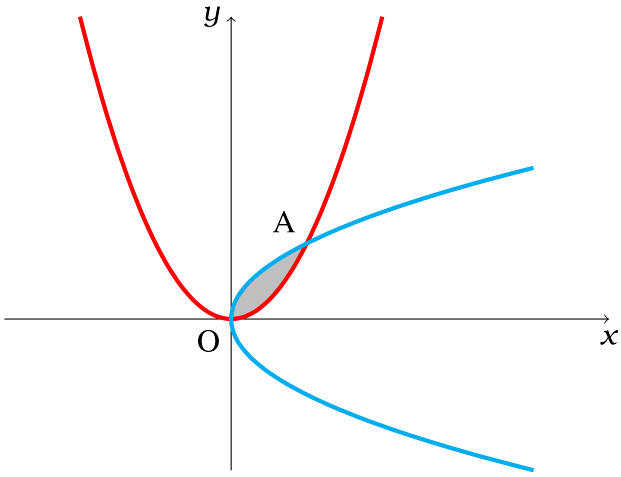
$$\frac{\sin^2 x}{p} = \frac{\cos^2 x}{q} \Leftrightarrow \cos^2 x = \frac{p}{p+q}$$

Since $\frac{p}{p+q} \in (0,1)$ *there exists an* $x \in \mathbb{R}$ *such that the equation has at least one root. The minimum follows.*

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3778`.

10 Let a be a positive real number. The parabolas defined by $y_1 = ax^2$ and $y_2^2 = ax$ intersect at the points O and A.



Prove that the area enclosed by the two curves is constant.

Solution. *First of all we note that*

$$\begin{aligned} y_1 = y_2 &\Leftrightarrow \left(ax^2\right)^2 = ax \\ &\Leftrightarrow a^2x^4 = ax \\ &\overset{a>0}{\Longleftrightarrow} ax^4 - x = 0 \\ &\Leftrightarrow x\left(ax^3 - 1\right) = 0 \\ &\Leftrightarrow \begin{cases} x &= 0 \\ x &= \sqrt[3]{\frac{1}{a}} \end{cases} \end{aligned}$$

Hence,

$$\begin{aligned} \mathrm{E}(\Omega) &= \int_0^{\sqrt[3]{1/a}} \left|ax^2 - \sqrt{ax}\right| \, \mathrm{d}x \\ &= \int_0^{\sqrt[3]{1/a}} \left(\sqrt{ax} - ax^2\right) \mathrm{d}x \\ &= \frac{2}{3} - \frac{1}{3} \\ &= \frac{1}{3} \end{aligned}$$

♦

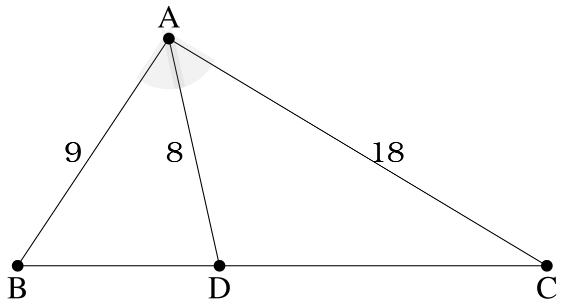
Exercise lies in `https://www.math.tolaso.com.gr/?p=3838`.

6

PART

JoM ... proposes

(1) Evaluate the area of the given triangle:



(2) Solve the system

$$(S):\begin{cases} x+\dfrac{1}{x^2+1} & = & y+\dfrac{1}{y^2+1} \\ x^2+2x\sqrt{y+\dfrac{1}{y}} & = & 8x-1 \end{cases}$$

(3) Let $\alpha, \beta, \gamma, \delta$ be four consecutive terms of a geometric sequence. Prove that

(a) $(\beta-\gamma)^2+(\gamma-\alpha)^2+(\delta-\beta)^2=(\alpha-\delta)^2$

(b) $(\alpha^2+\beta^2+\gamma^2)(\beta^2+\gamma^2+\delta^2)=(\alpha\beta+\beta\gamma+\gamma\delta)^2$

(4) Find the monotony of the function

$$f(x)=\ln x \ln \frac{x}{x-1}$$

(5) Evaluate the integral

$$\mathcal{J}=\int \frac{\sqrt{x^2+2}}{x^2+1}\,\mathrm{d}x$$

(6) Let $\mathcal{H}_n$ denote the n - th harmonic number. Evaluate the sum

$$\mathcal{S}=\sum_{n=1}^{\infty}\frac{\mathcal{H}_n}{\binom{n+k}{k}}$$

where $\mathbb{N} \ni k > 1$.

(7) Let $a \in (-1, 1)$. Prove that

$$\int_0^{\pi} \frac{\ln(1-2a\cos x+a^2)}{1-2a\cos x+a^2}\,\mathrm{d}x=\frac{2\pi\ln(1-a^2)}{1-a^2}$$

(8) Let $a \in \mathbb{R}$. Prove that

$$\int_0^{\infty} \ln\left(1+\frac{\cosh a}{\cosh x}\right)\mathrm{d}x=\frac{\pi^2}{8}+\frac{a^2}{2}$$

(9) Evaluate the integral

$$\mathcal{J} = \int_1^2 \frac{\mathrm{d}x}{x\left(1 + x\sqrt{x\sqrt[3]{x\sqrt[4]{x\cdots\sqrt[n]{x}}}}\right)}$$

(10) Prove that

$$\int_0^{\pi/2} \cos(\tan x - \cot x)\,\mathrm{d}x = \frac{\pi}{e^2}$$

(11) Evaluate the integral

$$\mathcal{J} = \int_0^1 \ln^2\left|\sqrt{x} - \sqrt{1-x}\right|\,\mathrm{d}x$$

(12) Evaluate the limit

$$\ell = \lim_{x\to 0}\frac{1}{x^3}\left(x - \int_0^{\int_0^{\int_0^{\int_0^{x} e^{-x_n^2}\,\mathrm{d}x_n}\cdot^{\cdot^{\cdot}} e^{-x_3^2}\,\mathrm{d}x_3} e^{-x_2^2}\,\mathrm{d}x_2} e^{-x_1^2}\,\mathrm{d}x_1\right)$$

(13) Let F_n denote the Fibonacci sequence defined as $F_0 = 0$, $F_1 = 1$ and

$$F_{n+1} = F_n + F_{n-1} \quad \text{for all } \; n \geq 1$$

Prove that

$$\prod_{n=0}^{\infty} \frac{\cosh F_{n+1} + i\sinh F_n}{\cosh F_{n+1} - i\sinh F_n} = i\left(\frac{e+i}{e-i}\right)^2$$

(14) Let $a \notin \mathbb{Z}$ and $a \neq \frac{1}{2}$. Prove that

$$\sum_{n=0}^{\infty} \frac{2}{\Gamma(a+n)\,\Gamma(a-n)} = \frac{2^{2a-2}}{\Gamma(2a-1)} + \frac{1}{\Gamma^2(a)}$$

7

PART

JoM ... study

Author: Tolaso

JoM ... studies functions and series

Lemma

All functions $f : \mathbb{R} \to \mathbb{R}$ that preserve convergent series are of the form

$$f(x) = ax \ , \ x \in (-\delta, \delta)$$

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then the following statements are equivalent.

1. f preserves convergent series , that is the series $\sum_{n=1}^{\infty} f(a_n)$ converges whenever the series $\sum_{n=1}^{\infty} a_n$ converges.
2. f is linear in a neighbourhood of zero , that is it is of the form $f(x) = ax, \ x \in (-\delta, \delta)$

The direction (2) $\implies$ (1) is obvious. We only need to prove the direction (1) $\implies$ (2).

Claim I

f is odd in a neighbourhood of zero.

Proof. Suppose on the contrary that this does not hold. Then there exists a sequence x_n such that $\lim x_n = 0$ that for each $n \in \mathbb{N}$ holds $y_n = f(x_n) + f(-x_n) \neq 0$. For all $n \in \mathbb{N}$ we choose a positive integer k_n such that $|y_n| > \frac{1}{k_n}$. We consider the sequence a_m that is defined piecewise as follows:

$$\underbrace{x_1, -x_1, x_1, -x_1, \ldots, x_1, -x_1}_{2k_1}, \underbrace{x_2, -x_2, x_2, -x_2, \ldots, x_2, -x_2}_{2k_2}, \ldots$$

$$\ldots, \underbrace{x_n, -x_n, x_n, -x_n, \ldots, x_n, -x_n}_{2k_n}, \ldots$$

that is its n-th part is consisted of $2k_n$ terms that are alternating equal to x_n and $-x_n$ respectively. Since $\lim x_n = 0$ this means that $\sum_{n=1}^{\infty} a_n = 0$. However from Cauchy's criterion the series $\sum_{n=1}^{\infty} f(a_n)$ does not converge because the sum of the terms of the n-th part is equal to $k_n|y_n| > 1$. Hence claim 1 is complete. ♦

Claim II

There exists a δ_1 such that for all $x, y \in (-\delta_1, \ \delta_1)$ holds:

$$f(x + y) = f(x) + f(y)$$

That is f is linear in a neighbourhood of zero.

Proof. Like above we choose k_n such that for all $n \in \mathbb{N}$ to hold $|z_n| > \dfrac{1}{k_n}$. We consider the sequence a_m that is defined piecewise as follows:
Its n-th part is consisted of $3k_n$ that are $x_n + y_n, -x_n, -y_n$ repeated k_n for'es. Since $\lim x_n = \lim y_n = 0$ we get $\sum_{n=1}^{\infty} a_n = 0$ but the series $\sum_{n=1}^{\infty} f(a_n)$ does not converge from Cauchy's criterion because the sum of the terms of its n-th part is equal to $k_n|z_n| > 1$. This proves claim 2. ♦

Claim III

There exist $\delta_2 > 0$ and a constant $C > 0$ such that forall $x \in (-\delta_2, \delta_2)$ to hold: $|f(x)| \leq Cx$

Proof. Suppose that the claim is false. Then there exists a sequence x_n such that forall $n \in \mathbb{N}$ to hold $|x_n| < \dfrac{1}{2^n}$ kai $|f(x_n)| \geq 2^n|x|$. We note that $x_n \neq 0$ for all $n \in \mathbb{N}$ (since from claim (1) holds $f(0) = 0$) .
Hence, for every $n \in \mathbb{N}$ we choose a positive integer k_n such that it holds:

$$\frac{1}{2^{n+1}} \leq k_n \, |x_n| \leq \frac{1}{2^n}$$

We consider the sequence a_m that is defined piecewise again as follows: its n-th part is consisted of $2k_n$ terms that are of the form:

$$\underbrace{x_n, x_n, \ldots, x_n}_{k_n}, \underbrace{-x_n, -x_n, \ldots, -x_n}_{k_n}$$

Again we have that $\sum_{n=1}^{\infty} a_n = 0$ but the series $\sum_{n=1}^{\infty} f(a_n)$ does not converge from Cauchy's criterion since the absolute value of the sum of the first k_n terms is:

$$k_n \, |f(x_n)| > k_n 2^n \, |x_n| \geq 2^n \cdot \frac{1}{2^{n+1}} = \frac{1}{2}$$

This proves claim 3.

♦

To finish things off we set $\delta = \frac{1}{2} \min \{\delta_1, \delta_2\}$ and we note that forall $x, y \in (-\delta, \delta)$ holds: $|x - y| < \min \{\delta_1, \delta_2\}$. Therefore from Claims (1) and (2) follows that:

$$|f\,(x) - f\,(y)| = |f\,(x - y)| \leq C\,|x - y|$$

Hence f is continuous, additive on $(-\delta, \delta)$ and linear.

♦

Author: Tolaso

JoM ... studies a particular differential equation

Lemma

We will find an explicit formula for $f : (0, +\infty) \to \mathbb{R}$ that is differentiable, $f(1) = \frac{1}{2}$ and

$$f'(x) + e^{f(x)} = x + \frac{1}{x} \quad \text{for all } \ x > 0$$

Proof. Let us consider the function $g(x) = f(x) - \ln x - \frac{x^2}{2}$ which is clearly differentiable. Hence,

$$\begin{aligned}
f'(x) + e^{f(x)} = x + \frac{1}{x} &\Leftrightarrow \left(g(x) + \ln x + \frac{x^2}{2}\right)' + e^{g(x)+\ln x+\frac{x^2}{2}} = x + \frac{1}{x} \\
&\Leftrightarrow g'(x) + \frac{1}{x} + x + xe^{g(x)}e^{x^2/2} = x + \frac{1}{x} \\
&\Leftrightarrow g'(x) + xe^{g(x)}e^{x^2/2} = 0 \\
&\Leftrightarrow g'(x) = -xe^{g(x)}e^{x^2/2} \\
&\Leftrightarrow e^{-g(x)}g'(x) = -xe^{x^2/2} \\
&\Leftrightarrow -e^{-g(x)}g'(x) = xe^{x^2/2} \\
&\Leftrightarrow \left(e^{-g(x)}\right)' = \left(e^{x^2/2}\right)' \\
&\Rightarrow e^{-g(x)} = e^{x^2/2} + c \\
&\xRightarrow{x=1 \Rightarrow c=1-\sqrt{e}} e^{-g(x)} = e^{x^2/2} + 1 - \sqrt{e} \\
&\Rightarrow g(x) = -\ln\left(e^{x^2/2} + 1 - \sqrt{e}\right)
\end{aligned}$$

Thus,

$$f(x) = \ln x + \frac{x^2}{2} - \ln\left(e^{x^2/2} + 1 - \sqrt{e}\right)$$

♦

Author: Tolaso

JoM ... studies an asymptotic formula

Theorem I

Let τ denote the divisor function. It holds that

$$\sum_{n\leq x}\tau(n) = x\ln x + (2\gamma - 1)x + \mathcal{O}\left(\sqrt{x}\right)$$

Proof. We start by providing a lemma:

Lemma (Dirichlet Hyperbola Method)

Let f, g, and h be multiplicative functions such that $f = g * h$, where $*$ denotes the convolution of g and h. Then

$$\begin{aligned}\sum_{n\leq x} f(n) &= \sum_{n\leq x}\sum_{ab=n} g(a)h(b)\\ &= \sum_{a\leq\sqrt{x}}\sum_{b\leq\frac{x}{a}} g(a)h(b) + \sum_{b\leq\sqrt{x}}\sum_{a\leq\frac{x}{b}} g(a)h(b)\\ &\quad - \sum_{a\leq\sqrt{x}}\sum_{b\leq\sqrt{x}} g(a)h(b)\end{aligned}$$

Note that, since $ab = n \leq x$, not both of a and b can be larger than $\sqrt{x}$. The Dirichlet hyperbola method follows from this fact as well as the inclusion-exclusion principle.

First of all we note that $\tau = 1 * 1$ hence,

$$\begin{aligned}\sum_{n\leq x}\tau(n) &= \sum_{a\leq\sqrt{x}}\sum_{b\leq\frac{x}{a}} 1 + \sum_{b\leq\sqrt{x}}\sum_{a\leq\frac{x}{b}} 1 - \sum_{a\leq\sqrt{x}}\sum_{b\leq\sqrt{x}} 1\\ &= \sum_{a\leq\sqrt{x}}\left(\frac{x}{a} + \mathcal{O}(1)\right) + \sum_{b\leq\sqrt{x}}\left(\frac{x}{b} + \mathcal{O}(1)\right) - \left(\sum_{a\leq\sqrt{x}} 1\right)\left(\sum_{b\leq\sqrt{x}} 1\right)\\ &= 2\sum_{c\leq\sqrt{x}}\left(\frac{x}{c} + \mathcal{O}(1)\right) - \left(\sum_{c\leq\sqrt{x}} 1\right)^2\\ &= 2x\sum_{c\leq\sqrt{x}}\frac{1}{c} + \mathcal{O}\left(\sum_{c\leq\sqrt{x}} 1\right) - \left(\sqrt{x} + \mathcal{O}(1)\right)^2\\ &= 2x\left(\ln\sqrt{x} + \gamma + \mathcal{O}\left(\frac{1}{\sqrt{x}}\right)\right) + \mathcal{O}\left(\sqrt{x}\right) - \left(x + \mathcal{O}\left(\sqrt{x}\right) + \mathcal{O}(1)\right)\\ &= 2x\left(\frac{\ln x}{2} + \gamma + \mathcal{O}\left(\frac{1}{\sqrt{x}}\right)\right) - x + \mathcal{O}\left(\sqrt{x}\right)\\ &= x\ln x + 2\gamma x + \mathcal{O}\left(\frac{x}{\sqrt{x}}\right) - x + \mathcal{O}\left(\sqrt{x}\right)\\ &= x\ln x + (2\gamma - 1)x + \mathcal{O}\left(\sqrt{x}\right)\end{aligned}$$

♦

Theorem II

Let τ denote the divisor function. It holds that

$$\sum_{n\le x}\frac{\tau(n)}{n}=\frac{\ln^2 x}{2}+2\gamma\ln x+\gamma^2-2\gamma_1+O\left(\frac{1}{\sqrt{x}}\right)$$

where γ_1 is the Stieltjes constant.

Proof. We are applying the hyperbola method again. First of all we note that

$$\sum_{n\le x}\frac{\tau(n)}{n}=\sum_{n\le x}\frac{1}{n}\sum_{ab=n}1=\sum_{ab\le x}\frac{1}{ab}$$

Rearranging based on the geometry of the hyperbola, this equals

$$2\sum_{a\le\sqrt{x}}\frac{1}{a}\sum_{b\le\frac{x}{a}}\frac{1}{b}-\sum_{a\le\sqrt{x}}\sum_{b\le\sqrt{x}}\frac{1}{ab}$$

Since $\sum_{b\le\frac{x}{a}}\frac{1}{b}=\ln\frac{x}{a}+\gamma+O\left(\frac{a}{x}\right)$ it follows that

$$2\sum_{a\le\sqrt{x}}\frac{1}{a}\sum_{b\le\frac{x}{a}}\frac{1}{b}=2\sum_{a\le\sqrt{x}}\frac{1}{a}\ln\left(\frac{x}{a}\right)+2\gamma\sum_{a\le\sqrt{x}}\frac{1}{a}+O\left(\frac{1}{x}\sum_{a\le\sqrt{x}}1\right)$$

so we have that

$$\begin{aligned}\sum_{n\le x}\frac{\tau(n)}{n}&=2\ln x\sum_{a\le\sqrt{x}}\frac{1}{a}-2\sum_{a\le\sqrt{x}}\frac{\ln a}{a}+2\gamma\left(\ln\sqrt{x}+\gamma\right)-\\&\quad-\left(\ln\sqrt{x}+\gamma+O\left(\frac{1}{x}\right)\right)^2+O\left(\frac{1}{\sqrt{x}}\right)\\&=\frac{3}{4}\ln^2 x+2\gamma\ln x-2\sum_{a\le\sqrt{x}}\frac{\ln a}{a}+\gamma^2+O\left(\frac{1}{\sqrt{x}}\right)\end{aligned}$$

By definition of the Stieltjies constants,

$$\sum_{a\le z}\frac{\ln a}{a}=\frac{\ln^2 z}{2}+\gamma_1+O\left(\frac{\ln z}{z}\right)$$

and the formula follows.

♦