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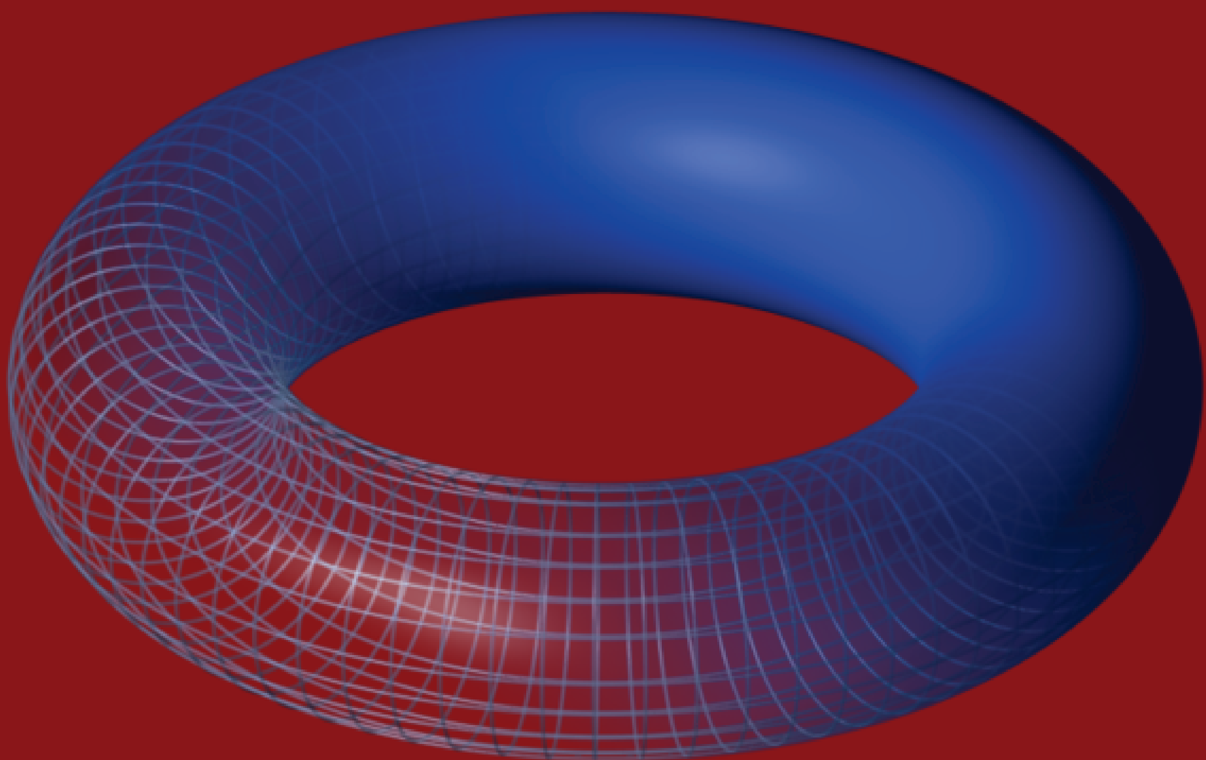

TOLASO NETWORK
2020

The JoM Journal

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The JoM Journal is an electronic mathematical journal which aims at giving the chance to the readers, and the editor himself, to work on interesting mathematical problems or find information about various mathematical topics. The problems presented here are basically a collection of the problems posted on the JoM Blog (hosted at math.tolaso.com.gr) . The level of the topics is undergraduate and beyond. However, there is a section dedicated to inequalities and general mathematics sometimes including mathematical competitions. The JoM journal is consisted of 6 parts:

- Algebra
- Calculus
- Real Analysis
- Inequalities
- JoM ... proposes
- JoM ... study

The JoM ... proposes column contains problems that extend the ideas already seen in the previous 4 columns. The JoM study, on the other hand, studies several mathematical concepts. Examples are included whenever necessary. At the end of this part the reader will find problems to exercise himself.

If you want to submit an article at the JoM ... study please contact the author at tolaso@tolaso.com.gr.

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Part 1

Algebra

(1) Let $A = \begin{pmatrix} -2 & 4 & 3 \\ 0 & 0 & 0 \\ -1 & 5 & 2 \end{pmatrix}$. Prove that $A^{593} - 2A^{15} + A = \mathbb{O}_{3\times 3}$.

Solution. *The characteristic polynomial of* A *is* $p(x) = x - x^3$. *This in return means* $A = A^3$ *and* $A^3 = A^5$. *Thus,*

$$\begin{aligned} A^{593} - 2A^{15} + A &= A^{591} \cdot A^2 - 2\left(A^3\right)^5 + A \\ &= \left(A^3\right)^{197} \cdot A^2 - 2A^5 + A \\ &= A^{197} \cdot A^2 - 2A^3 + A \\ &= A^{199} - 2A + A \\ &= A^{198} \cdot A - A \\ &= \left(A^3\right)^{66} \cdot A - A \\ &= A^{66} \cdot A - A \\ &= \left(A^3\right)^{22} \cdot A - A \\ &= A^{22} \cdot A - A \\ &= A^{23} - A \\ &= \left(A^3\right)^{7} \cdot A^2 - A \\ &= A^7 \cdot A^2 - A \\ &= A^9 - A \\ &= \left(A^3\right)^3 - A \\ &= A^3 - A \\ &= A - A \\ &= \mathbb{O}_{3\times 3} \end{aligned}$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=2905`.

(2) Let $x > 0$ and $A \in \mathbb{R}^{2\times 2}$ such that $\det\left(A^2 + x\mathbb{I}_{2\times 2}\right) = 0$. Prove that

$$\det\left(A^2 + A + x\mathbb{I}_{2\times 2}\right) = x$$

Solution. *Note that*

$$\det\left(A^2 + x\mathbb{I}_{2\times 2}\right) = \det\left(A + i\sqrt{x}\mathbb{I}_{2\times 2}\right)\det\left(A - i\sqrt{x}\mathbb{I}_{2\times 2}\right)$$

Since A *is real, its complex eigenvalues come in conjugate pairs. Thus, in this case we conclude that* A *has eigenvalues* $\pm i\sqrt{x}$.

Now, if λ *is an eigenvalue of* A, *then* $\lambda^2 + \lambda + x$ *is an eigenvalue of* $A^2 + A + x\mathbb{I}_{2\times 2}$. *Thus, the matrix* $A^2 + A + x\mathbb{I}_{2\times 2}$ *has eigenvalues* $(i\sqrt{x})^2 + i\sqrt{x} + x = i\sqrt{x}$ *and* $(-i\sqrt{x})^2 - i\sqrt{x} + x = -i\sqrt{x}$.

Now, $\det\left(A^2 + A + x\mathbb{I}_{2\times 2}\right)$ *is the product of these eigenvalues, which is to say*

$$\det\left(A^2 + A + x\mathbb{I}_{2\times 2}\right) = (i\sqrt{x})(-i\sqrt{x}) = x$$

as desired.

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=2923`.

(3) Suppose that $A \in \mathcal{M}_n(\mathbb{C})$ has this property that if λ is an eigenvalue of A then $-\lambda$ is not an eigenvalue of A. Show that $AX = XA$ if and only if $A^2X = XA^2$ for any $X \in \mathcal{M}_n(\mathbb{C})$. In other words the centralizer of A equals the centralizer of A^2. [1]

Solution. *It is clear that $AX = XA$ implies $A^2X = XA^2$ for any $X \in \mathcal{M}_n(\mathbb{C})$. Now suppose that $A^2X = XA^2$ for some $X \in \mathcal{M}_n(\mathbb{C})$ and set $Y = AX - XA$. We want to prove that $Y = 0$. We have*

$$\begin{aligned} AY + YA &= A(AX - XA) + (AX - XA)A \\ &= A^2X - XA^2 \\ &= 0 \end{aligned}$$

and so $AY = -YA$. It now follows that $A^kY = (-1)^kYA^k$ for any integer $k \geq 0$ and thus for any $\lambda \in \mathbb{C}$ and any integer $m \geq 0$ we have

$$\begin{aligned} (A + \lambda\mathbb{I})^m Y &= \sum_{k=0}^{m} \binom{m}{k} \lambda^{m-k} A^k Y \\ &= \sum_{k=0}^{m} \binom{m}{k} (-1)^k \lambda^{m-k} Y A^k \\ &= (-1)^m Y \sum_{k=0}^{m} \binom{m}{k} (-\lambda)^{m-k} A^k \\ &= (-1)^m Y (A - \lambda\mathbb{I}_n)^m \quad (*) \end{aligned}$$

where $\mathbb{I}$ is the identity matrix. Now let v be a generalized eigenvector corresponding to an eigenvalue λ of A. Then $(A - \lambda\mathbb{I}_n)^m v = 0$ for some integer m and thus, by $(*)$ we have $(A + \lambda\mathbb{I}_n)^m Yv = 0$. Therefore, since we are assuming that $-\lambda$ is not an eigenvalue of A, we must have $Yv = 0$. So, since every element of $\mathbb{C}^n$ is a linear combination of some generalized eigenvectors of A, we get $Yu = 0$ for all $u \in \mathbb{C}^n$, i.e. $Y = 0$ and hence $AX = XA$.

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=2960`.

(4) Let $A \in \mathcal{M}_n(\mathbb{C})$ such that

$$A \det A + A^* \det A^* = i\,(A + A^*)$$

Prove that $\det A = 0$ if n is odd.

[1] Exercise lies in `https://bit.ly/2VFfUCP`.

Solution. *Let* $\det A = z$. *Then*

$$zA + \bar{z}A^* = i(A + A^*)$$

Taking conjugate transpose we also have that

$$\bar{z}A^* + zA = -i(A^* + A)$$

Hence $A + A^* = 0$. *However it also holds* $zA + \bar{z}A^* = 0$. *Combiming these two we get that*

$$(z - \bar{z})A = 0$$

If $A = 0$ *we are done. Otherwise* z *is real. In that case we have*

$$z = \det A = \det(-A^*) = (-1)^n \det A^* = -\bar{z} = -z$$

since n *is odd. Hence* $\det A = z = 0$ *as wanted.*

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3068`.

5 Consider the real numbers x_i, x_j for $1 \leq i, j \leq 3$. Prove that

$$\mathcal{D} = \begin{vmatrix} \sin(x_1 + y_1) & \sin(x_1 + y_2) & \sin(x_1 + y_3) \\ \sin(x_2 + y_1) & \sin(x_2 + y_2) & \sin(x_2 + y_3) \\ \sin(x_3 + y_1) & \sin(x_3 + y_2) & \sin(x_3 + y_3) \end{vmatrix} = 0$$

Solution. *Using the identity* $\sin(a+b) = \sin a \cos b + \sin b \cos a$ *in combination with* $\det AB = \det A \det B$ *we have:*

$$\mathcal{D} = \begin{vmatrix} \sin x_1 & \cos x_1 & 0 \\ \sin x_2 & \cos x_2 & 0 \\ \sin x_3 & \cos x_3 & 0 \end{vmatrix} \cdot \begin{vmatrix} \cos y_1 & \cos y_2 & \cos y_3 \\ \sin y_1 & \sin y_2 & \sin y_3 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3218`.

PART 2

Calculus

1 Let $\alpha, \beta > 0$ such that $\alpha \neq \beta$. Prove that

$$\int_0^\infty \frac{1-x^2}{(\alpha x+\beta)^2} \frac{\ln(1+x)}{(\beta x+\alpha)^2}\, \mathrm{d}x = \frac{1}{\alpha\beta\left(\alpha^2-\beta^2\right)} \ln \frac{\beta}{\alpha}$$

Solution. *Let $\mathcal{J}$ denote the given integral. Applying the change of variables $x \mapsto \frac{1}{x}$ we get*

$$\begin{aligned}
\mathcal{J} &= \int_0^\infty \frac{(1-x^2)\ln(1+x)}{(\alpha x+\beta)^2(\alpha+\beta x)^2}\, \mathrm{d}x \\
&\overset{x\to\frac{1}{x}}{=\!=} -\int_0^\infty \frac{(1-x^2)(\ln(1+x)-\ln x)}{(\alpha x+\beta)^2(\alpha+\beta x)^2}\, \mathrm{d}x
\end{aligned}$$

Thus,

$$2\mathcal{J} = \int_0^\infty \frac{(1-x^2)\ln x}{(\alpha x+\beta)^2(\alpha+\beta x)^2}\, \mathrm{d}x = \frac{1}{\alpha^2-\beta^2}\int_0^\infty \frac{\ln x}{(\alpha x+\beta)^2}\, \mathrm{d}x - \frac{1}{\alpha^2-\beta^2}\int_0^\infty \frac{\ln x}{(\alpha+\beta x)^2}\, \mathrm{d}x$$

However, we note that

$$\int_0^\infty \frac{\ln x}{(\alpha+\beta x)^2}\, \mathrm{d}x \overset{x\to\frac{1}{x}}{=\!=} -\int_0^\infty \frac{\ln x}{(\alpha x+\beta)^2}\, \mathrm{d}x$$

Hence,

$$\begin{aligned}
\mathcal{J} &= \frac{1}{\alpha^2-\beta^2}\int_0^\infty \frac{\ln x}{(\alpha x+\beta)^2}\, \mathrm{d}x \\
&\overset{\alpha x\to bt}{=\!=} \frac{1}{\alpha\beta(\alpha^2-b^2)}\int_0^\infty \frac{\ln\left(\frac{\beta}{\alpha}t\right)}{(1+t)^2}\, \mathrm{d}t \\
&\overset{t\mapsto 1/t}{=\!=} \frac{1}{\alpha\beta\left(\alpha^2-\beta^2\right)}\int_0^\infty \frac{\ln\left(\frac{\beta}{\alpha}\frac{1}{t}\right)}{(1+t)^2}\, \mathrm{d}t \\
&= \frac{1}{\alpha\beta\left(\alpha^2-\beta^2\right)}\cdot\frac{1}{2}\left(\int_0^\infty \frac{\ln\left(\frac{\beta}{\alpha}t\right)}{(1+t)^2}\, \mathrm{d}t + \int_0^\infty \frac{\ln\left(\frac{\beta}{\alpha}\frac{1}{t}\right)}{(1+t)^2}\, \mathrm{d}t\right) \\
&= \frac{1}{\alpha\beta\left(\alpha^2-\beta^2\right)}\cdot\frac{1}{2}\cdot\int_0^\infty \frac{2\ln\frac{\beta}{\alpha}}{(1+t)^2}\, \mathrm{d}t \\
&= \frac{1}{\alpha\beta\left(\alpha^2-\beta^2\right)}\ln\frac{\beta}{\alpha}\cancelto{1}{\int_0^\infty \frac{\mathrm{d}t}{(1+t)^2}} \\
&= \frac{1}{\alpha\beta\left(\alpha^2-\beta^2\right)}\ln\frac{\beta}{\alpha}
\end{aligned}$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=2821`.

2 Let Γ denote the Euler's Gamma function. Prove that

$$\int_{-\infty}^\infty \frac{\mathrm{d}x}{\Gamma(\alpha+x)\Gamma(\beta-x)} = \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}$$

where $\Re\mathfrak{e}(\alpha+\beta) > 1$.

Solution. *We are proving the more general result*

$$\int_{-\infty}^{\infty} \frac{e^{inx}}{\Gamma(\alpha+x)\Gamma(\beta-x)}\,\mathrm{d}x = \frac{\left(2\cos\frac{n}{2}\right)^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} e^{in(\beta-\alpha)/2}$$

First of all we note that

$$\begin{aligned}
\frac{1}{\Gamma(\alpha+x)\Gamma(\beta-x)} &= \frac{1}{(\alpha+x-1)!\,(\beta-x-1)!} \\
&= \frac{1}{\Gamma(\alpha+\beta-1)} \frac{(\alpha+\beta-2)!}{(\alpha+x-1)!\,(\beta-x-1)!} \\
&= \frac{1}{\Gamma(\alpha+\beta-1)} \binom{\alpha+\beta-2}{\alpha+x-1}
\end{aligned}$$

Thus,

$$\int_{-\infty}^{\infty} \frac{e^{inx}}{\Gamma(\alpha+x)\Gamma(\beta-x)}\,\mathrm{d}x = \frac{1}{\Gamma(\alpha+\beta-1)} \int_{-\infty}^{\infty} \binom{\alpha+\beta-2}{\alpha+x-1} e^{inx}\,\mathrm{d}x$$

Lemma

Let γ be any circle circumference. It holds that:

$$\binom{n}{k} = \frac{1}{2\pi i} \oint_{\gamma} \frac{(1+z)^n}{z^{k+1}}\,\mathrm{d}z$$

Proof. The complex function $f(z) = \dfrac{(1+z)^n}{z^{k+1}}$ is meromorphic in $\mathbb{C}$ and has a pole at $z_0 = 0$ of order $k+1$. However,

$$\begin{aligned}
\mathfrak{Res}\,(f;0) &= \frac{1}{k!} \lim_{z\to 0} \left(z^{k+1} f(z)\right)^{(k)} \\
&= \frac{1}{k!} \lim_{z\to 0} \left((1+z)^n\right)^{(k)} \\
&= \frac{n(n-1)\cdots(n-k+1)}{k!} \\
&= \binom{n}{k}
\end{aligned}$$

This completes the proof. ♦

Thus,

$$\begin{aligned}
\int_{-\infty}^{\infty} \binom{\alpha+\beta-2}{\alpha+x-1} e^{inx}\,\mathrm{d}x &= \int_{-\infty}^{\infty} \left(\oint_{|z|=1} \frac{(1+z)^{\alpha+\beta-2}}{z^{\alpha+x}} \frac{\mathrm{d}z}{2\pi i} \right) e^{inx}\,\mathrm{d}x \\
&= -i \oint_{|z|=1} \frac{(1+z)^{\alpha+\beta-2}}{z^{\alpha}} \left(\int_{-\infty}^{\infty} e^{i(n-\arg z)x} \frac{\mathrm{d}x}{2\pi} \right) \mathrm{d}z
\end{aligned}$$

$$\begin{aligned}
&= -i \oint_{|z|=1} \frac{(1+z)^{\alpha+\beta-2}}{z^{\alpha}} \, \delta(n - \arg z) \, \mathrm{d}z \\
&\overset{z=e^{i\vartheta}\,,\,|\vartheta|<\pi}{=\!=\!=\!=\!=} -i \int_{-\pi}^{\pi} \frac{\left(1+e^{i\vartheta}\right)^{\alpha+\beta-2}}{e^{i\alpha\vartheta}} \, \delta(n-\vartheta) \, e^{i\vartheta} i \, \mathrm{d}\vartheta \\
&= \left(1+e^{in}\right)^{\alpha+\beta-2} e^{i(1-\alpha)n} \\
&= \left(2\cos\frac{n}{2}\right)^{\alpha+\beta-2} e^{in(\beta-\alpha)/2}
\end{aligned}$$

The result follows.

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=2816`.

(3) Let J_n denote the Bessel function of the first kind. [2] Prove that

$$\sum_{n=-\infty}^{\infty} |J_n(z)|^2 = 1$$

Solution. *The Jacobi – Anger expansion tells us that*

$$e^{iz\sin\vartheta} = \sum_{n=-\infty}^{\infty} J_n(z) e^{in\vartheta} \tag{1}$$

Hence by Parseval's Theorem it follows that

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} |J_n(z)|^2 &= \frac{1}{2\pi} \int_0^{2\pi} e^{iz\sin\vartheta} e^{iz\sin(-\vartheta)} \, \mathrm{d}\vartheta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\vartheta \\
&= 1
\end{aligned}$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=2804`.

(4) Evaluate the series

$$\mathcal{S} = \sum_{n=1}^{\infty} \arcsin\left(\frac{\sqrt{n} - \sqrt{n-1}}{\sqrt{n}\sqrt{n+1}}\right)$$

Solution. *The series telescopes since,*

$$\arcsin\left(\frac{\sqrt{n} - \sqrt{n-1}}{\sqrt{n}\sqrt{n+1}}\right) = \arcsin\sqrt{\frac{n}{n+1}} - \arcsin\sqrt{\frac{n-1}{n}}$$

Hence the limit equals $\frac{\pi}{2}$.

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=2844`.

[2] `https://bit.ly/2yuHM4s`

5 Evaluate the integral

$$\mathcal{J} = \int_0^{2\pi} \ln\left(\sin x + \sqrt{1+\sin^2 x}\right) \, \mathrm{d}x$$

Solution. *let* f *be an odd and periodic function of period* $\mathrm{T} = 2\pi$. *Then,*

$$\begin{aligned}
\int_0^{2\pi} \ln\left(f(x) + \sqrt{1+f^2(x)}\right) \, \mathrm{d}x &= \int_{-\pi}^{\pi} \ln\left(f(x) + \sqrt{1+f^2(x)}\right) \, \mathrm{d}x \\
&= \int_{-\pi}^{0} \ln\left(f(x) + \sqrt{1+f^2(x)}\right) \, \mathrm{d}x + \\
&\quad + \int_0^{\pi} \ln\left(f(x) + \sqrt{1+f^2(x)}\right) \, \mathrm{d}x \\
&= \int_0^{\pi} \ln\left(-f(x) + \sqrt{1+f^2(x)}\right) \, \mathrm{d}x + \\
&\quad + \int_0^{\pi} \ln\left(f(x) + \sqrt{1+f^2(x)}\right) \, \mathrm{d}x \\
&= \int_0^{\pi} \left(\ln\left(-f(x) + \sqrt{1+f^2(x)}\right) + \right. \\
&\quad \left. + \ln\left(f(x) + \sqrt{1+f^2(x)}\right) \right) \mathrm{d}x \\
&= \int_0^{\pi} \ln 1 \, \mathrm{d}x \\
&= 0
\end{aligned}$$

Let $f(x) = \sin x$. *It follows that* $\mathcal{J} = 0$.

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=2828`.

6 Let $0 < \alpha < \beta$. Evaluate the integral:

$$\mathcal{J} = \int_\alpha^\beta \frac{\ln x}{(x+\alpha)(x+\beta)} \, \mathrm{d}x$$

Solution. *We have successively:*

$$\begin{aligned}
\mathcal{J} &= \int_\alpha^\beta \frac{\ln x}{(x+\alpha)(x+\beta)} \, \mathrm{d}x \\
&\overset{x \mapsto \alpha\beta/x}{=\!=} \int_\alpha^\beta \frac{\ln \alpha\beta - \ln x}{(x+\alpha)(x+\beta)} \, \mathrm{d}x \\
&= \frac{1}{2} \int_\alpha^\beta \frac{\ln \alpha\beta}{(x+\alpha)(x+\beta)} \, \mathrm{d}x \\
&= \frac{\ln \alpha\beta}{\beta - \alpha} \ln\left(\frac{(\alpha+\beta)^2}{4\alpha\beta}\right)
\end{aligned}$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=2915`.

7 Let $0 < a < b$. Evaluate the integral

$$\mathcal{J} = \int_a^b \frac{e^{x/a} - e^{b/x}}{\sqrt{abx + x^3}} \, \mathrm{d}x$$

Solution. *The key substitution is $x \mapsto \frac{ab}{u}$. Applying it we see that*

$$\begin{aligned}
\sqrt{abx + x^3} &\overset{x \mapsto ab/u}{=\!=\!=} \sqrt{\frac{a^2b^2}{u} + \frac{a^3b^3}{u^3}} \\
&= \sqrt{\frac{a^2b^2u^2}{u^3} + \frac{a^3b^3}{u^3}} \\
&= \sqrt{\frac{a^2b^2\left(u^2 + ab\right)}{u^2 \cdot u}} \\
&= \frac{ab}{u} \sqrt{\frac{u^2 + ab}{u}}
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathcal{J} &= \int_a^b \frac{e^{x/a} - e^{b/x}}{\sqrt{abx + x^3}} \, \mathrm{d}x \\
&\overset{x = ab/u}{=\!=\!=} ab \int_a^b \frac{1}{u^2} \cdot \left(e^{b/u} - e^{u/a}\right) \cdot \frac{u}{ab} \cdot \frac{\sqrt{u}}{\sqrt{u^2 + ab}} \, \mathrm{d}u \\
&= \int_a^b \frac{e^{b/u} - e^{u/a}}{\sqrt{abu + u^3}} \, \mathrm{d}u \\
&= -\int_a^b \frac{e^{u/a} - e^{b/u}}{\sqrt{abu + u^3}} \, \mathrm{d}u \\
&= -\mathcal{J}
\end{aligned}$$

Thus $\mathcal{J} = 0$.

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=2909`.

8 Evaluate the integral

$$\mathcal{J} = \int_0^1 (x \ln x)^{2020} \, \mathrm{d}x$$

Solution. *Recall the idenity*

$$\ln \mathbf{x} = \lim_{\mathbf{n} \to +\infty} \mathbf{n} \left(\mathbf{x}^{1/\mathbf{n}} - 1\right)$$

thus

$$\begin{aligned}
\int_0^1 (x \ln x)^{2020} \, \mathrm{d}x &= \lim_{n \to +\infty} n^{2020} \int_0^1 \left(x^{2020} \left(x^{1/n} - 1\right)^{2020}\right) \mathrm{d}x \\
&\overset{u = x^{1/n}}{=\!=\!=} \lim_{n \to +\infty} n^{2021} \int_0^1 u^{2021n - 1} (1 - u)^{2020} \, \mathrm{d}u
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n \to +\infty} n^{2021} \int_0^1 u^{2021n-1} (1-u)^{2021-1} \, \mathrm{d}u \\
&= \lim_{n \to +\infty} n^{2021} \mathrm{B}\left(2021n, 2021\right) \\
&= \lim_{n \to +\infty} n^{2021} \frac{\Gamma\left(2021n\right) \Gamma\left(2021\right)}{\Gamma\left(2021n+2021\right)} \\
&= \Gamma\left(2021\right) \lim_{n \to +\infty} n^{2021} \frac{\Gamma\left(2021n\right)}{\Gamma\left(2021n+2021\right)}
\end{aligned}
$$

Using Gautschi's Inequality it follows that

$$
n^{2021} \left(2021n-1\right)^{1-2022} < \frac{n^{2021} \Gamma\left(2021n-1+1\right)}{\Gamma\left(2021n-1+2022\right)} < n^{2021} \left(2021n\right)^{1-2022}
$$

and hence the integral equals

$$
\mathcal{J} = \frac{\Gamma(2021)}{2021^{2021}}
$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=2952`.

9 Let $a, b > 0$. Prove that:

$$
\int_a^b \frac{\mathrm{d}t}{t \sqrt{(t-a)(b-t)}} = \frac{\pi}{\sqrt{ab}}
$$

Solution. *We're applying the change of variables* $t \mapsto a\cos^2 \vartheta + b \sin^2 \vartheta$ *and thus,*

$$
\begin{aligned}
\int_a^b \frac{\mathrm{d}t}{t \sqrt{(t-a)(b-t)}} &= \int_0^{\pi/2} \frac{2\left(b-a\right) \sin \vartheta \cos \vartheta}{\left(a \cos^2 \vartheta + b \sin^2 \vartheta\right) \sqrt{\left(b-a\right)^2 \sin^2 \vartheta \cos^2 \vartheta}} \, \mathrm{d}\vartheta \\
&= 2 \int_0^{\pi/2} \frac{\mathrm{d}\vartheta}{a \cos^2 \vartheta + b \sin^2 \vartheta} \\
&= 2 \int_0^{\pi/2} \frac{\mathrm{d}\vartheta}{\cos^2 \vartheta \left(a + b \tan^2 \vartheta\right)} \\
&= 2 \int_0^{\pi/2} \frac{\sec^2 \vartheta}{a + b \tan^2 \vartheta} \, \mathrm{d}\vartheta \\
&\overset{y=\tan \vartheta}{=\!=} 2 \int_0^{\infty} \frac{\mathrm{d}y}{a + by^2} \\
&= \frac{\pi}{\sqrt{ab}}
\end{aligned}
$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=2997`.

10 Prove that

$$
\int_{-1}^1 \frac{\mathrm{d}x}{\sqrt{1+x} + \sqrt{1-x} + 2} = 4\sqrt{2} - \pi - 2
$$

Solution. *First of all we note that*

$$\left(\sqrt{1+x}+\sqrt{1-x}\right)^2 = 2\left(1+\sqrt{1-x^2}\right)$$

Hence,

$$\begin{aligned}
\int_{-1}^{1} \frac{\mathrm{d}x}{\sqrt{1+x}+\sqrt{1-x}+2} &= \int_{-1}^{1} \frac{\mathrm{d}x}{\sqrt{2\left(1+\sqrt{1-x^2}\right)}+2} \\
&\overset{x=\sin u}{=\!=\!=} \int_{-\pi/2}^{\pi/2} \frac{\cos u}{\sqrt{2\left(1+\cos u\right)}+2}\,\mathrm{d}u \\
&= \int_{-\pi/2}^{\pi/2} \frac{\cos u}{\sqrt{4\cos^2\frac{u}{2}}+2}\,\mathrm{d}u \\
&= \int_{-\pi/2}^{\pi/2} \frac{\cos u}{2\cos\frac{u}{2}+2}\,\mathrm{d}u \\
&= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \frac{\cos u}{1+\cos\frac{u}{2}}\,\mathrm{d}u \\
&= \frac{1}{4}\int_{-\pi/2}^{\pi/2} \frac{\cos u}{\cos^2\frac{u}{4}}\,\mathrm{d}u \\
&= \cancelto{0}{\frac{1}{4}\left[4\tan\frac{u}{4}\cos u\right]_{-\pi/2}^{\pi/2}} + \\
&\quad + \int_{-\pi/2}^{\pi/2} \tan\frac{u}{4}\sin u\,\mathrm{d}u
\end{aligned}$$

To finish things off we evaluate the final integral. For example:

$$\begin{aligned}
\int_{-\pi/2}^{\pi/2} \tan\frac{u}{4}\sin u\,\mathrm{d}u &\overset{y=u/4}{=\!=\!=} 4\int_{-\pi/8}^{\pi/8} \tan y \sin 4y\,\mathrm{d}y \\
&= 4\int_{-\pi/8}^{\pi/8} \tan y\left(4\sin y\cos^3 y - 4\sin^3 y\cos y\right)\,\mathrm{d}y \\
&= 4\int_{-\pi/8}^{\pi/8} \left(4\sin^2 y\cos^2 y - 4\sin^4 y\right)\,\mathrm{d}y \\
&= 16\int_{-\pi/8}^{\pi/8} (\sin y\cos y)^2\,\mathrm{d}y - 16\int_{-\pi/8}^{\pi/8} \sin^4 y\,\mathrm{d}y \\
&= \left(\frac{\pi}{2}-1\right) + \left(-1+4\sqrt{2}-\frac{3\pi}{2}\right) \\
&= 4\sqrt{2}-\pi-2
\end{aligned}$$

in view of the identities

$$\sin^4 x = \frac{1}{8}\left(3+\cos 4x - 4\cos 2x\right)$$

and

$$\begin{aligned}\int_{-\pi/8}^{\pi/8} \sin^2 x \cos^2 x \, \mathrm{d}x &= \int_{-\pi/8}^{\pi/8} (\sin x \cos x)^2 \, \mathrm{d}x \\ &= \int_{-\pi/8}^{\pi/8} \left(\frac{\sin 2x}{2}\right)^2 \mathrm{d}x \\ &= \frac{1}{4}\int_{-\pi/8}^{\pi/8} \sin^2 2x \, \mathrm{d}x \\ &= \frac{1}{8}\int_{-\pi/8}^{\pi/8} (1-\cos 4x) \, \mathrm{d}x \\ &= \frac{1}{8}\left(\frac{\pi}{4}-\frac{1}{2}\right) \\ &= \frac{\pi}{32}-\frac{1}{16}\end{aligned}$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=2938`.

11 Prove that

$$\int_0^\infty \frac{\sin \sqrt{x^2+1}\cos x}{\sqrt{x^2+1}} \, \mathrm{d}x = \frac{\pi}{4}$$

Solution. *Let* $\mathcal{J}$ *be the integral. Note that*

$$2\sin \sqrt{x^2+1}\cos x = \sin\left(\sqrt{1+x^2}-x\right) + \sin\left(\sqrt{1+x^2}+x\right)$$

and hence:

$$2\mathcal{J} = \int_0^\infty \frac{\sin\left(\sqrt{1+x^2}-x\right)}{\sqrt{1+x^2}} \, \mathrm{d}x + \int_0^\infty \frac{\sin\left(\sqrt{1+x^2}+x\right)}{\sqrt{1+x^2}} \, \mathrm{d}x$$

For the integral $\int_0^\infty \frac{\sin\left(\sqrt{1+x^2}-x\right)}{\sqrt{1+x^2}} \, \mathrm{d}x$ *we apply the substitution* $t \mapsto \sqrt{1+x^2}-x$. *Then,* $x = \frac{1-t^2}{2t}$ *and*
and

$$\int_0^\infty \frac{\sin\left(\sqrt{1+x^2}-x\right)}{\sqrt{1+x^2}} \, \mathrm{d}x = \int_0^1 \frac{\sin t}{t} \, \mathrm{d}t \tag{1}$$

and similarly by applying the change of variables $t \mapsto \sqrt{1+x^2}+x$ *at the second integral we get that*

$$\int_0^\infty \frac{\sin\left(\sqrt{1+x^2}\right)}{\sqrt{1+x^2}} \, \mathrm{d}x = \int_1^\infty \frac{\sin t}{t} \, \mathrm{d}t \tag{2}$$

Adding equations (1), (2) *we get that*

$$2\mathcal{J} = \int_0^1 \frac{\sin t}{t} \, \mathrm{d}t + \int_1^\infty \frac{\sin t}{t} \, \mathrm{d}t$$

$$= \int_0^\infty \frac{\sin t}{t}\,\mathrm{d}t$$
$$= \frac{\pi}{2}$$

and the result follows.

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3006`.

12 Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_{2n\pi}^{\infty} \frac{\sin z}{z}\,\mathrm{d}z = \pi - \frac{\pi \ln 2\pi}{2}$$

Solution. *We have successively:*

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n} \int_{2\pi n}^{\infty} \frac{\sin z}{z}\,\mathrm{d}z &= \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} \frac{\sin nt}{t+2\pi}\,\mathrm{d}t \\
&= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{n} \int_0^{2\pi} \frac{\sin nt}{t+2m\pi}\,\mathrm{d}t \\
&= \sum_{m=1}^{\infty} \int_0^{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} \frac{\sin nt}{t+2m\pi}\,\mathrm{d}t \\
&= \sum_{m=1}^{\infty} \int_0^{2\pi} \frac{\pi - t}{2\,(t+2m\pi)}\,\mathrm{d}t \\
&= \pi \sum_{n=1}^{\infty} \left[\left(1+\frac{1}{2}\right) \ln\left(1+\frac{1}{n}\right) - 1 \right] \\
&= \pi \ln \left[\lim_{N\to+\infty} e^{-N} \prod_{n=1}^{N} \left(\frac{n+1}{n}\right)^{n+1/2} \right] \\
&= \pi \ln \left(\lim_{N\to+\infty} \frac{\sqrt{N+1}\,(N+1)^N\, e^{-N}}{N!} \right) \\
&= \pi \ln \left(\frac{e}{\sqrt{2\pi}} \right) \\
&= \pi - \frac{\pi \ln 2\pi}{2}
\end{aligned}$$

since for $x \in (0, 2\pi)$ it holds that

$$\sum_{n=1}^{\infty} \frac{\sin nt}{n} = \frac{\pi - t}{2}$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3168`.

13 Prove that

$$\sum_{n=1}^{\infty} \left(\psi^{(1)}(n)\right)^2 = 3\zeta(3)$$

Solution. *Since* $\psi^{(1)}(n) = \sum_{k=n}^{\infty} \frac{1}{k^2}$ *we have successively:*

$$\begin{aligned}
\sum_{n=1}^{\infty} \left(\psi^{(1)}(n)\right)^2 &= \sum_{n=1}^{\infty} \sum_{j=n}^{\infty} \frac{1}{j^2} \sum_{k=n}^{\infty} \frac{1}{k^2} \\
&= \sum_{n=1}^{\infty} \left(\sum_{j=n}^{\infty} \frac{1}{j^4} + 2 \sum_{j=n}^{\infty} \sum_{m=1}^{\infty} \frac{1}{j^2} \frac{1}{(j+m)^2} \right) \\
&= \sum_{j=1}^{\infty} \sum_{n=1}^{j} \frac{1}{j^4} + 2 \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{j} \frac{1}{j^2} \frac{1}{(j+m)^2} \\
&= \sum_{j=1}^{\infty} \frac{1}{j^3} + 2 \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{j(j+m)^2} \\
&= \zeta(3) + 2 \sum_{n=1}^{\infty} \frac{\mathcal{H}_{n-1}}{n^2} \\
&= \zeta(3) - 2\zeta(3) + 2 \sum_{n=1}^{\infty} \frac{\mathcal{H}_n}{n^2} \\
&= \zeta(3) - 2\zeta(3) + 4\zeta(3) \\
&= 3\zeta(3)
\end{aligned}$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3157`.

14 Prove that

$$\int_0^{\infty} \frac{\sin^2 \tan x}{x^2} \, dx = \frac{\pi}{2}$$

Solution. *We state the following lemmata:*

Lemma 1

Let $x \in \mathbb{R} \setminus \mathbb{Z}$. *It holds that*

$$\pi \cot \pi x = \frac{1}{x} + \sum_{n=1}^{\infty} \left(\frac{1}{x+n} - \frac{1}{x-n} \right) = \lim_{N \to +\infty} \sum_{n=-N}^{N} \frac{1}{x+n}$$

Proof. A standard proof can be found through Fourier Series. One can expand in Fourier series the function $f(x) = \cos ax$, $a \notin \mathbb{Z}$, $|x| \leq \pi$. Another way to prove the identity is to begin from the Weierstrass product , that is $\sin \pi x = \pi x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right)$. Taking log on both sides we have that

$$\begin{aligned}
\ln \sin \pi x &= \ln \pi x + \sum_{n=1}^{\infty} \ln \left(1 - \frac{x^2}{n^2}\right) \\
&= \ln \pi x + \sum_{n=1}^{\infty} \left[\ln \left(1 - \frac{x}{n}\right) - \ln \left(1 + \frac{x}{n}\right) \right]
\end{aligned}$$

Differentiating we have

$$\frac{\pi \cos \pi x}{\sin \pi x} = \frac{1}{x} + \sum_{n=1}^{\infty} \left(\frac{1}{x+n} - \frac{1}{x-n} \right) = \lim_{N \to +\infty} \sum_{n=-N}^{N} \frac{1}{x+n}$$

♦

Lemma 2

It holds that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^2} = \frac{\pi^2}{\sin^2 \pi z}$$

Proof. Just differentiate the above identity. ♦

Applying the above lemmata , we have for the initial integral that

$$\begin{aligned}
\int_0^\infty \frac{\sin^2 \tan x}{x^2} \, dx &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin^2 \tan x}{x^2} \, dx \\
&= \frac{1}{2} \sum_{n=-\infty}^{\infty} \int_{-\pi/2}^{\pi/2} \frac{\sin^2 \tan(x+n\pi)}{(x+n\pi)^2} \, dx \\
&= \frac{1}{2} \int_{-\pi/2}^{\pi/2} \sin^2 \tan x \left(\sum_{n=-\infty}^{\infty} \frac{1}{(x+n\pi)^2} \right) dx \\
&= \frac{1}{2} \int_{-\pi/2}^{\pi/2} \frac{\sin^2 \tan x}{\sin^2 x} \, dx \\
&= \int_0^{\pi/2} \frac{\sin^2 \cot x}{\cos^2 x} \, dx \\
&= \int_0^\infty \sin^2 \frac{1}{y} \, dy \\
&= \frac{\pi}{2}
\end{aligned}$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3098`.

15 Evaluate the integral

$$\mathcal{J} = \int_0^\pi \arctan^2 \left(\frac{\sin x}{2 + \cos x} \right) dx$$

Solution. *First of all we note that for* $x \in (0, \pi)$

$$\begin{aligned}
\arctan \left(\frac{\sin x}{2 + \cos x} \right) &= \Im\mathfrak{m} \ln \left(2 + e^{ix}\right) \\
&= \Im\mathfrak{m} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n 2^n} e^{inx}
\end{aligned}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n2^n} \sin nx$$

Hence by Parseval we get that

$$\int_0^{\pi} \arctan^2\left(\frac{\sin x}{2+\cos x}\right) \mathrm{d}x = \frac{\pi}{2} \cdot \sum_{n=1}^{\infty} \frac{1}{n^2 4^n} = \frac{\pi}{2} \cdot \mathrm{Li}_2\left(\frac{1}{4}\right)$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3213`.

16 Let $a \in \mathbb{R}$. Evaluate the integral

$$\mathcal{J} = \int_0^a \ln\left(1 + \tan a \tan x\right) \mathrm{d}x$$

Solution. *We have successively:*

$$\begin{aligned}
\mathcal{J} &= \int_0^a \ln\left(1 + \tan a \tan x\right) \mathrm{d}x \\
&\overset{x \mapsto a-x}{=\!=\!=} \int_0^a \ln\left(1 + \tan a \tan\left(a - x\right)\right) \mathrm{d}x \\
&= \int_0^a \ln\left(1 + \tan a \cdot \frac{\tan a - \tan x}{1 + \tan a \tan x}\right) \mathrm{d}x \\
&= \int_0^a \ln\left(1 + \frac{\tan^2 a - \tan a \tan x}{1 + \tan a \tan x}\right) \mathrm{d}x \\
&= \int_0^a \ln\left(\frac{1 + \tan a \tan x + \tan^2 a - \tan a \tan x}{1 + \tan a \tan x}\right) \mathrm{d}x \\
&= \int_0^a \ln \frac{1 + \tan^2 a}{1 + \tan a \tan x} \mathrm{d}x \\
&= \int_0^a \ln\left(1 + \tan^2 a\right) \mathrm{d}x - \mathcal{J} \\
&= a \ln\left(1 + \tan^2 a\right) - \mathcal{J}
\end{aligned}$$

Hence

$$\mathcal{J} = \frac{a \ln\left(1 + \tan^2 a\right)}{2}$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3278`.

PART 3

Analysis

1 Evaluate the limit:

$$\ell = \lim_{x\to+\infty} \int_{\frac{1}{x+1}}^{\frac{1}{x}} \cot t^2 \, dt$$

Solution. *Recalling the Taylor expansion of* tan *around* $x = 0$ *we have that*

$$\cot x^2 = \frac{1}{x^2} + \mathcal{O}\left(x^2\right)$$

Thus,

$$\begin{aligned}
\int_{\frac{1}{x+1}}^{\frac{1}{x}} \cot t^2 \, dt &= \int_{\frac{1}{x+1}}^{\frac{1}{x}} \left(\cot t^2 - \frac{1}{t^2} + \frac{1}{t^2}\right) dt \\
&= \int_{\frac{1}{x+1}}^{\frac{1}{x}} \left(\cot t^2 - \frac{1}{t^2}\right) dt + \int_{\frac{1}{x+1}}^{\frac{1}{x}} \frac{dt}{t^2} \\
&= \mathcal{O}\left(\int_{\frac{1}{x+1}}^{\frac{1}{x}} t^2\right) + 1 \\
&= \mathcal{O}\left(\frac{1}{x^3} - \frac{1}{(x+1)^3}\right) + 1 \\
&= 1 + \mathcal{O}\left(\frac{1}{x^4}\right)
\end{aligned}$$

The limit follows to be 1*.*

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=2884`.

2 Let $f : [0, 1] \to \mathbb{R}$ be a function such that

$$f(x^2) + \sqrt{x} f\left(x^2\sqrt{x}\right) = e^x \quad \text{forall} \quad x \in [0, 1]$$

Evaluate the integral $\int_0^1 f(x)\, dx$.

Solution. *We multiply the given equation by* x*. Thus,*

$$\begin{aligned}
f\left(x^2\right) + \sqrt{x} f\left(x^2\sqrt{x}\right) = e^x &\Leftrightarrow x f\left(x^2\right) + x\sqrt{x} f\left(x^2\sqrt{x}\right) = x e^x \\
&\Leftrightarrow \int_0^1 x f\left(x^2\right) dx + \int_0^1 x\sqrt{x} f\left(x^2\sqrt{x}\right) dx = \int_0^1 x e^x \, dx
\end{aligned}$$

Let $\mathcal{J}_1 = \int_0^1 x f\left(x^2\right) dx$ *and* $\mathcal{J}_2 = \int_0^1 x\sqrt{x} f\left(x^2\sqrt{x}\right) dx$*.*
We now deal with the first integral:

$$\int_0^1 x f\left(x^2\right) dx \overset{u=x^2}{=\!=} \frac{1}{2}\int_0^1 f(u)\, du$$

As for the second integral we have:

$$\int_0^1 x\sqrt{x} f\left(x^2\sqrt{x}\right) \mathrm{d}x \overset{u=x^2\sqrt{x}}{=\!=\!=} \frac{2}{5}\int_0^1 f(u)\,\mathrm{d}u$$

Hence,

$$\int_0^1 f(x)\,\mathrm{d}x = \frac{10}{9}$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=2874`.

3 Prove that

$$\int_2^{e+1} \frac{\mathrm{d}t}{\ln t} < e$$

Solution. *We have successively:*

$$\begin{aligned} \ln x \leq x-1 &\Rightarrow \ln\frac{1}{x} \leq \frac{1}{x} - 1 \\ &\Rightarrow -\ln x \leq \frac{1}{x} - 1 \\ &\Rightarrow \ln x \geq 1 - \frac{1}{x} \\ &\Rightarrow \frac{1}{\ln x} \leq \frac{1}{1-\frac{1}{x}} \\ &\Rightarrow \int_2^{e+1} \frac{\mathrm{d}t}{\ln t} < \int_2^{e+1} \frac{\mathrm{d}t}{1-\frac{1}{t}} \\ &\Rightarrow \int_2^{\mathbf{e+1}} \frac{\mathrm{d}\mathbf{t}}{\ln \mathbf{t}} < \mathbf{e} \end{aligned}$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=2870`.

4 Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence such that $a_1 = 3$ and

$$a_{n+1} = a_n^2 - 2 \quad \text{forall} \quad n \geq 2$$

Evaluate the sum

$$\mathcal{S} = \sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{1}{a_k}$$

Solution. *Let us consider the sequence*

$$b_n = \frac{a_n - \sqrt{a_n^2 - 4}}{2}$$

and observe that

$$\begin{aligned} b_{n+1} &= \frac{a_{n+1} - \sqrt{a_{n+1}^2 - 4}}{2} \\ &= \frac{a_n^2 - 2 - \sqrt{(a_n^2-2)^2 - 4}}{2} \\ &= \frac{a_n\left(a_n - \sqrt{a_n^2 - 4}\right) - 2}{2} \\ &= a_n b_n - 1 \end{aligned}$$

This in return means,

$$b_n = \frac{1}{a_n} + \frac{b_{n+1}}{a_n}$$

Thus,

$$\begin{aligned} b_1 &= \frac{1}{a_1} + \frac{b_2}{a_1} \\ &= \frac{1}{a_1} + \frac{1}{a_1}\left(\frac{1}{a_2} + \frac{b_3}{a_2}\right) \\ &= \frac{1}{a_1} + \frac{1}{a_1 a_2}\left(\frac{1}{a_3} + \frac{b_4}{a_3}\right) \\ &= \cdots \end{aligned}$$

which is the desired result. Therefore,

$$\mathcal{S} = \sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{1}{y_k} = b_1 = \frac{3 - \sqrt{5}}{2}$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=2945`.

5 Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a function satisfying

$$|f(a+b) - f(a)| \leq \frac{b}{a}$$

for all positive real numbers a and b. Prove that

$$|f(1) - f(x)| \leq |\ln x| \quad \text{for all} \quad x > 0$$

Solution. *For starters, let us assume that* $x > 1$. *Dividing the interval* $(1, x)$ *into* n *subintervals each of length* h *so that* $h = \frac{x-1}{n}$. *Thus,*

$$\begin{aligned} |f(1) - f(x)| &= \left| \sum_{k=0}^{n-1} f(1 + h(k+1)) - f(1 + kh) \right| \\ &\leq \sum_{k=0}^{n-1} |f(1 + h(k+1)) - f(1 + kh)| \end{aligned}$$

$$= \sum_{k=0}^{n-1} |f((1+kh)+h) - f(1+kh)|$$

The inequality $|f(a+b) - f(a)| \leq \frac{b}{a}$ *implies that*

$$|f((1+kh)+h) - f(1+kh)| \leq \frac{h}{1+kh}$$

Hence,

$$|f(1) - f(x)| \leq \sum_{k=0}^{n-1} \frac{h}{1+kh}$$

The limit $\displaystyle\sum_{k=0}^{n-1} \frac{h}{1+kh}$ *exists and equals to* $\ln x$*. Hence , the inequality is proved for* $x > 1$.

Now, assume that $x < 1$*. Dividing the interval* $(x, 1)$ *into* n *subintervals each of length* h *so that* $h = \frac{1-x}{n}$*. Thus,*

$$\begin{aligned} |f(1) - f(x)| &= \left| \sum_{k=0}^{n-1} f(1 - h(k+1)) - f(1-kh) \right| \\ &\leq \sum_{k=0}^{n-1} |f(1-h(k+1)) - f(1-kh)| \\ &= \sum_{k=0}^{n-1} |f((1-kh)-h) - f(1-kh)| \end{aligned}$$

The inequality $|f(a+b) - f(a)| \leq \frac{b}{a}$ *implies that*

$$|f((1-kh)-h) - f(1-kh)| \leq \frac{h}{1-kh}$$

Hence,

$$|f(1) - f(x)| \leq \sum_{k=0}^{n-1} \frac{h}{1+kh}$$

The limit $\displaystyle\sum_{k=0}^{n-1} \frac{h}{1-kh}$ *exists and equals to* $-\ln x$*. Hence , the inequality is also proved for* $x < 1$*. This completes the proof!*

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=2934`.

6 Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that

$$\int_0^1 f(x)\,\mathrm{d}x = \int_0^1 x f(x)\,\mathrm{d}x = 0 \tag{1}$$

Prove that there exist $a, b \in [0, 1]$ such that $a < b$ and $f(a) = f(b) = 0$.

Solution. *The existence of* a, b *follows from Mean Value Theorem of Integrals. To prove that* $a \neq b$ *we proceed as follows:*

We are given that $\int_0^1 (ax+b)f(x)\,\mathrm{d}x = 0$*. Suppose* f *is not identically zero for otherwise the result is trivial. The condition* $\int_0^1 f(x)\,\mathrm{d}x = 0$ *implies there is at least one sign-changing root, say at* m *(that is, the function has different signs after passing through* m*, or more formally,* $f(m+\epsilon)f(m-\epsilon) < 0$ *for all sufficiently small* $\epsilon > 0$*.) Suppose this is the only sign-changing root. Then* $(x-m)f(x)$ *does not change signs and is not identically* 0 *either, so* $\int_0^1 (x-m)f(x)\,\mathrm{d}x \neq 0$*, contradicting the first statement. Thus there are at least* 2 *distinct sign-changing roots.*

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3043`.

(7) Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that

$$\int_0^1 f(x)\,\mathrm{d}x = \kappa = \int_0^1 xf(x)\,\mathrm{d}x \tag{1}$$

Prove that $\int_0^1 f^2(x)\,\mathrm{d}x \geq 4\kappa^2$.

Solution. *We have successively:*

$$\begin{aligned}
\int_0^1 f^2(x)\,\mathrm{d}x &= \int_0^1 (3x-1)^2\,\mathrm{d}x \int_0^1 f(x)^2\,\mathrm{d}x \\
&\geq \left(\int_0^1 (3x-1)f(x)\,\mathrm{d}x \right)^2 \\
&= (3k-k)^2 \\
&= 4k^2
\end{aligned}$$

Since f *is continuous equality holds only if* $f(x) = C(3x-1)$ *for some constant* C*. From the data we get* $C = 2k$*.*

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3071`.

(8) Consider the branch of $f(z) = \sqrt{z^2-1}$ which is defined outside the segment $[-1,1]$ and which coincides with the positive square root $\sqrt{x^2-1}$ for $x > 1$. Let $R > 1$ then evaluate the contour integral:

$$\oint_{|z|=R} \frac{\mathrm{d}z}{\sqrt{z^2-1}}$$

Solution. *It is a classic case of residue at infinity. Subbing* $z \mapsto \frac{1}{z}$ *the counterclockwise contour integral rotates the northern pole of the Riemannian sphere to the southern one and the contour integral is transformed to a clockwise one. Hence:*

$$\oint_{|z|=R} \frac{\mathrm{d}z}{\sqrt{z^2-1}} = -2\pi i \mathfrak{Res}_{z=\infty} \frac{1}{\sqrt{z^2-1}}$$

$$= -2\pi i \operatorname*{\mathfrak{Res}}_{w=0} \frac{-1}{w^2\sqrt{w^{-2}-1}}$$
$$= 2\pi i \operatorname*{\mathfrak{Res}}_{w=0} \frac{1}{w\sqrt{1-w^2}}$$
$$= 2\pi i \lim_{w\to 0} \frac{1}{\sqrt{1-w^2}}$$
$$= 2\pi i$$

The equality $w\sqrt{w^{-2}-1} = \sqrt{1-w^2}$ ***does hold*** *for all* $|w| < 1$ *if we take the standard branch* $\sqrt{1-w^2} = \exp\left(\frac{1}{2}\mathrm{Log}\left(1-w^2\right)\right)$ *, otherwise it is not that obvious why this holds, since we are dealing with a multi-valued function.*

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3188`.

(9) Find the derivative of

$$f(x) = \frac{\sqrt{1+2x}\cdot\sqrt[4]{1+4x}\cdot\sqrt[6]{1+6x}\cdot\dots\cdot\sqrt[100]{1+100x}}{\sqrt[3]{1+3x}\cdot\sqrt[5]{1+5x}\cdot\sqrt[7]{1+7x}\cdot\dots\cdot\sqrt[101]{1+101x}}$$

at 0.

Solution. *The domain of* f *is* $\mathcal{A}_f = \left[-\frac{1}{101}, +\infty\right)$. *Let us now consider the logarithmic function* f. *Hence,*

$$g(x) = \ln f(x) = \sum_{n=2}^{101} \frac{(-1)^n}{n} \ln(1+nx)$$

Differentiating we get

$$g'(x) = \frac{f'(x)}{f(x)} = \sum_{n=2}^{101} \frac{(-1)^n}{1+nx}$$

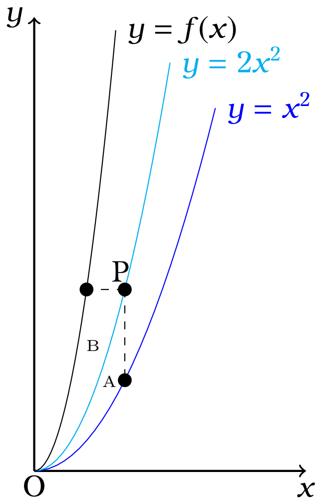
Setting $x = 0$ *we get*

$$g'(0) = \frac{f'(0)}{f(0)} = \sum_{n=2}^{101} (-1)^n = 0 \Leftrightarrow f'(0) = 0$$

♦

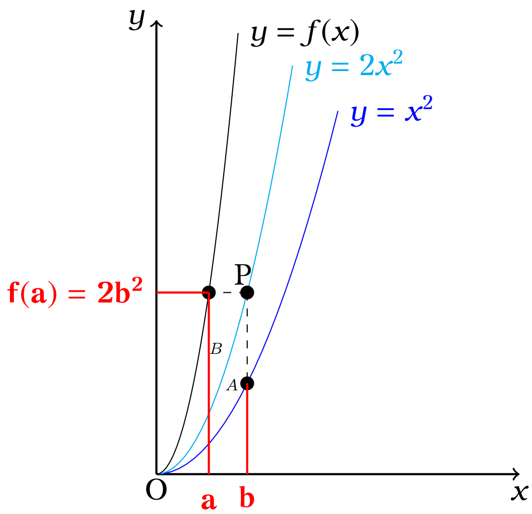
Exercise lies in `https://www.math.tolaso.com.gr/?p=3220`.

(10) In the following figure the function f is continuous and $1-1$. For every point P on the curve $y = 2x^2$ the areas of A and B are equal.



You are asked to find an explicit formula for f.

Solution. *We are expanding the above figure so that it looks like this;*



The area A *is given by*

$$\mathrm{A} = \int_0^{\mathbf{b}} 2x^2 \, \mathrm{d}x - \int_0^{\mathbf{b}} x^2 \, \mathrm{d}x = \int_0^{\mathbf{b}} x^2 \, \mathrm{d}x = \frac{\mathbf{b}^3}{3}$$

On the other hand we have that

$$\mathbf{b} f(a) = \int_0^{\mathbf{b}} x^2 \, \mathrm{d}x + \mathrm{A} + \mathrm{B} + \left(a f(a) - \int_0^{\mathbf{a}} f(x) \, \mathrm{d}x \right)$$

$$\overset{\mathrm{A}=\mathrm{B}\ ,\ f(a)=2b^2}{=\!=\!=\!=} \int_0^{\mathbf{a}} f(x) \, \mathrm{d}x = 2\mathbf{a}\mathbf{b}^2 - \mathbf{b}^3$$

Differentiating with respect to $\mathbf{b}$*and* $\mathbf{a}(\mathbf{b})$ *dependent on* $\mathbf{b}$ *we get that* $\mathbf{b} = \frac{3a}{4}$. *Hence,*

$$f(x) = \frac{32x^2}{9}$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3259`.

11 Let a_n be a sequence such that $a_n \geq 0$ and $0 < \sum_{n=1}^{\infty} a_n < +\infty$. Prove that the limit $\lim_{x \to +\infty} \sum_{n=1}^{\infty} a_n \sin \frac{x}{n}$ does not exist.

Solution. *Set* $f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{x}{n}$. *From the initial conditions we have that* f *is well defined and continuous. If we set* $x = 2\pi k!$ *then*

$$f(x) = \sum_{n=k+1}^{\infty} a_n \sin \frac{x}{n}$$

and of course $|f(x)| \leq \sum_{k=n+1}^{\infty} a_n$. *From this we conclude that if the limit exists , it is* 0. *Suppose that* n_0 *is the least natural such that* $a_{n_0} > 0$. *Then there exists natural* $m > n_0$ *such that* $l > m$ *to hold:*

$$\sum_{n=l}^{\infty} a_n < \frac{a_{n_0}}{200 n_0}$$

For $x = 2\pi k! + \frac{\pi}{4}$ *it holds*

$$\begin{aligned} f(x) &= \sum_{n=n_0}^{k} a_n \sin \frac{\pi}{4n} + \sum_{n=k+1}^{\infty} a_n \sin \frac{x}{n} \\ &> a_{n_0} \cdot \frac{2}{\pi} \cdot \frac{\pi}{4n_0} + \sum_{n=k+1}^{\infty} a_n \sin \frac{x}{n} \\ &> a_{n_0} \cdot \frac{2}{\pi} \cdot \frac{\pi}{4n_0} - \sum_{n=k+1}^{\infty} a_n \\ &> a_{n_0} \cdot \frac{2}{\pi} \cdot \frac{\pi}{4n_0} - \frac{a_{n_0}}{200 n_0} \end{aligned}$$

for $k > m$. *This shows that the limit does not exist.*

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3313`.

12 Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function. Suppose that $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of positive terms such that $\lim_{n\to+\infty} a_n = 0$. Prove that

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n} + a_n\right) = \int_0^1 f(x)\,\mathrm{d}x$$

Solution. *Let* $\epsilon > 0$. *Since* f *is uniformly continuous there exists* $\delta > 0$ *such that for each* x, y *with the property* $|x-y| < \delta$ *it holds* $|f(x)-f(y)| < \epsilon$. *Suppose* n_0 *such that for* $n \geq n_0$ *it holds* $|a_n| < \delta$. *For such* n *we have*

$$f\left(\frac{k}{n}\right) - \epsilon < f\left(\frac{k}{n} + a_n\right) < f\left(\frac{k}{n}\right) + \epsilon \tag{1}$$

Summing (1) *we get*

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) - \epsilon < \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n} + a_n\right) < \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) + \epsilon$$

Since $\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) \xrightarrow{n\to+\infty} \int_0^1 f(x)\,\mathrm{d}x$ *the result follows.*

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3318`.

13 Give an example of a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that for each $x \in \mathbb{Q}$ it holds $f(x) \in \mathbb{Q}$ and $f'(x) \notin \mathbb{Q}$.

Solution. *Let us define*

$$F(x) = x(1-x)(1+x) \quad , \quad -1 \leq x \leq 1$$

and extend it periodically to all of $\mathbb{R}$. *Specifically* $F(0) = F(-1) = F(1)$ *and hence* F *has zeroes all the integers. We also note that* F *is differentiable and the derivative is given by* $F'(x) = 1 - 3x^2$, $-1 \leq 1 \leq 1$. *The only thing we have to check is the differentiability of the function at the odd integers. But that is quite easy since*

$$\lim_{x\to 1^-} F'(x) = -2 = \lim_{x\to -1^+} F'(x)$$

Let us define the function under question as

$$f(x) = \sum_{n=0}^{\infty} \frac{F\,(n!x)}{(n!)^2} \tag{1}$$

The above sum converges uniformly because the nominator is bounded. We can also differentiate term by term since the series of f' *also converges uniformly.*

All that remains now is to prove that f *, the way it is defined , has the requested properties. Let* q *be a rational number. For all* n *that are greater or equal to the denominator of* q , $n!q$ *is rational. Hence ,* $F(n!q)$. *This in return means that the sum in* (1) *for rational* q *is a* ***finite rational sum*** *, hence rational as we wanted.*

All that remains now is to prove that $f'(q) \notin \mathbb{Q}$. *What we are using is a slightly variant of the well known proof that* e *is irrational. Indeed, let* $f(q) = \frac{P}{Q}$ *where* P, Q *are integers. Without loss of generality , let us suppose that they are positive. Thus, for a suitable large natural number* $R > Q$ *we have that* $R!f(q)$ *is an integer and*

$$\begin{aligned}
0 &< R!\left(f(q) - \sum_{n=0}^{R} \frac{F'(n!x)}{(n!)^2}\right) \\
&< \sum_{n=R+1}^{\infty} \frac{R!}{n!} \\
&= \frac{1}{R+1} + \frac{1}{(R+1)(R+2)} + \frac{1}{(R+1)(R+2)(R+3)} + \dots
\end{aligned}$$

$$
\begin{aligned}
&< \frac{1}{R+1} + \frac{1}{(R+1)^2} + \frac{1}{(R+1)^3} + \ldots \\
&= \frac{1}{R} \\
&< 1
\end{aligned}
$$

However , the last is a contradiction since the LHS is an integer and the RHS is less than 1. *This completes the proof.*

♦

Exercise lies in https://www.math.tolaso.com.gr/?p=3327.

4

PART

General Mathematics - Inequalities

1 For a rational number x that equals $\frac{a}{b}$ in lowest terms , let $f(x) = ab$. Prove that:

$$\sum_{x \in \mathbb{Q}^+} \frac{1}{f^2(x)} = \frac{5}{2}$$

Solution. *First of all we note that*

$$F(s) = \sum_{x \in \mathbb{Q}^+} \frac{1}{f^s(x)} = \sum_{\substack{a,b=1 \\ \gcd(a,b)=1}}^{\infty} \frac{1}{(ab)^s}$$

Moreover for $s > 1$ we have that

$$\begin{aligned}
\zeta^2(s) &= \left(\sum_{a=1}^{\infty} \frac{1}{a^s} \right)^2 \\
&= \sum_{a,b=1}^{\infty} \frac{1}{(ab)^s} \\
&= \sum_{d=1}^{\infty} \sum_{\substack{a,b=1 \\ \gcd(a,b)=d}}^{\infty} \frac{1}{(ab)^s} \\
&= \sum_{d=1}^{\infty} \frac{1}{d^{2s}} \sum_{\substack{a,b=1 \\ \gcd(a,b)=1}}^{\infty} \frac{1}{(ab)^s} \\
&= \zeta(2s) F(s)
\end{aligned}$$

Hence for $s = 2$ we have that

$$F(2) = \frac{5}{2}$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=2931`.

2 Let μ denote the Möbius function and $\lfloor \cdot \rfloor$ denote the floor function. Prove that:

$$n! = \prod_{j=1}^{\infty} \prod_{i=1}^{\infty} \left(\left\lfloor \frac{n}{ij} \right\rfloor ! \right)^{\mu(i)}$$

Solution. *The RHS equals*

$$\begin{aligned}
\prod_{j=1}^{\infty} \prod_{i=1}^{\infty} \left(\left\lfloor \frac{n}{ij} \right\rfloor ! \right)^{\mu(i)} &= \prod_{k=1}^{n} \prod_{i \mid k} \left(\left\lfloor \frac{n}{k} \right\rfloor ! \right)^{\mu(i)} \\
&= \prod_{k=1}^{n} \left(\left\lfloor \frac{n}{k} \right\rfloor ! \right)^{\sum_{i \mid k} \mu(i)} \\
&= n!
\end{aligned}$$

since $\sum_{i|k} \mu(i) = 0$ *for* $k > 1$.

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=2950`.

3 Let $a, b, c \in \mathbb{Q}$ such that $a \neq b \neq c \neq a$. Prove that

$$\mathcal{A} = \sqrt{\frac{1}{(a-b)^2} + \frac{1}{(b-c)^2} + \frac{1}{(c-a)^2}}$$

is rational.

Solution. *Setting* $a-b = x$,$b-c = y$ *and* $c-a = z$ *we note that* $x+y+z = 0$. *Hence,*

$$\begin{aligned}\sqrt{\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}} &= \sqrt{\frac{x^2y^2 + y^2z^2 + z^2x^2}{x^2y^2z^2}} \\ &= \sqrt{\frac{(xy+yz+zx)^2 - 2xyz\overbrace{(x+y+z)}^{0}}{x^2y^2z^2}} \\ &= \sqrt{\left(\frac{xy+yz+zx}{xyz}\right)^2}\end{aligned}$$

The result follows.

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=2958`.

4 Let $\kappa > 0$. Prove that

$$\sum_{n=1}^{\infty} \frac{n \mod \kappa}{n^2+n} = \ln \kappa$$

Solution. *Writing* $n = qk + r$ *with* $r \in \{0, 1, 2, \ldots, k-1\}$ *we want to calculate*

$$\sum_{q=0}^{\infty} \sum_{r=1}^{k-1} \frac{r}{(qk+r)^2 + (qk+r)} = \sum_{q=0}^{\infty} \sum_{r=1}^{k-1} \left(\frac{r}{qk+r} - \frac{r}{qk+r+1}\right)$$

Telescopically we have that

$$\begin{aligned}\sum_{r=1}^{k-1} \left(\frac{r}{qk+r} - \frac{r}{qk+r+1}\right) &= \frac{1}{qk+1} + \frac{1}{qk+2} + \cdots + \frac{1}{qk+k-1} - \frac{k-1}{qk+k} \\ &= \frac{1}{qk+1} + \frac{1}{qk+2} + \cdots + \frac{1}{qk+k} - \frac{1}{q+1}\end{aligned}$$

On the other hand we also have that

$$\sum_{q=0}^{N}\left(\frac{1}{qk+1}+\frac{1}{qk+2}+\cdots+\frac{1}{qk+k}\right)=H_{(N+1)k}=\ln(N+1)k+\gamma+o(1)$$

and

$$\sum_{q=0}^{N}\frac{1}{q+1}=H_{N+1}=\ln(N+1)+\gamma+o(1)$$

This proves the result.

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3063`.

5 Let a, b, c be three positive real numbers such that $abc = 1$. Prove that:

$$\frac{\Gamma(a)}{1+a+ab}+\frac{\Gamma(b)}{1+b+bc}+\frac{\Gamma(c)}{1+c+ca}\geq 1$$

where Γ is Euler's Gamma function.

Solution. *We can rewrite the inequality as*

$$\begin{aligned}\sum\frac{\Gamma(a)}{1+a+ab}&=\frac{\Gamma(a)}{1+a+ab}+\frac{a}{a}\frac{\Gamma(b)}{1+b+bc}+\frac{ab}{ab}\frac{\Gamma(c)}{1+c+ca}\\&=\frac{\Gamma(a)}{1+a+ab}+\frac{a\Gamma(b)}{a+ab+abc}+\frac{ab\Gamma(c)}{ab+abc+abca}\\&=\frac{\Gamma(a)}{1+a+ab}+\frac{a\Gamma(b)}{1+a+ab}+\frac{ab\Gamma(c)}{1+a+ab}\\&=\frac{\Gamma(a)+a\Gamma(b)+ab\Gamma(c)}{1+a+ab}\\&\geq\Gamma\left(\frac{1+a+ab}{1+a+ab}\right)\\&=1\end{aligned}$$

since Γ *is convex.*

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3244`.

6 Let ϕ denote the golden ratio , μ the Möbius function and φ Euler's totient function. Prove that:

$$\prod_{n=1}^{\infty}\left(1-\frac{1}{\phi^n}\right)^{\frac{\mu(n)-\varphi(n)}{n}}=e$$

Solution. *We are using the following facts:*

$$\sum_{d|m}\varphi(d)=m \tag{1}$$

$$\sum_{d|m} \mu(d) = \begin{cases} 1 & , \quad m = 1 \\ 0 & , \quad m > 1 \end{cases} \tag{2}$$

Hence,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\mu(n) - \varphi(n)}{n} \ln\left(1 - \frac{1}{\varphi^n}\right) &= -\sum_{n=1}^{\infty} \frac{\mu(n) - \varphi(n)}{n} \sum_{k=1}^{\infty} \frac{1}{k\varphi^{kn}} \\
&= \sum_{m=1}^{\infty} \frac{1}{m\varphi^m} \sum_{d|m} (\varphi(d) - \mu(d)) \\
&= \sum_{m=2}^{\infty} \frac{1}{\varphi^m} \\
&= \frac{1}{\varphi^2} \frac{1}{1 - \frac{1}{\varphi}} \\
&= \frac{1}{\varphi^2 - \varphi} \\
&= 1
\end{aligned}$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3235`.

7 Evaluate the sum

$$\mathcal{S} = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{d|n} \frac{d}{n + d^2}$$

Solution. *The sum converges absolutely , so we can switch the order of summation; hence:*

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n} \sum_{d|n} \frac{d}{n + d^2} &= \sum_{d=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{dk} \frac{d}{dk + d^2} \\
&= \sum_{d=1}^{\infty} \frac{1}{d} \sum_{k=1}^{\infty} \frac{1}{k(d + k)} \\
&= \sum_{d=1}^{\infty} \frac{1}{d^2} \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{d + k}\right) \\
&= \sum_{d=1}^{\infty} \frac{\mathcal{H}_d}{d^2} \\
&= 2\zeta(3)
\end{aligned}$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3249`.

8 Let μ denote the Möbius function and φ the Euler's totient function. Prove that

$$\frac{n}{\varphi(n)} = \sum_{d|n} \frac{\mu^2(d)}{\varphi(d)}$$

Solution. *Since both functions are multiplicative it suffices to prove the identity for prime numbers. Hence for $n = p^k$ we have*

$$\begin{aligned}\sum_{d \mid p^k} \frac{\mu^2(d)}{\varphi(d)} &= 1 + \frac{1}{\varphi(p)} \\ &= \frac{p}{p-1} \\ &= \frac{p^k}{p^{k-1}(p-1)} \\ &= \frac{p^k}{\varphi(p^k)}\end{aligned}$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3322`.

9 Prove that there does not exist a rational function f with real coefficients such that

$$f\left(\frac{x^2}{x+1}\right) = \mathrm{P}(x)$$

where $\mathrm{P}(x) \in \mathbb{R}[x]$ is a non constant polynomial.

Solution. *Since polynomials are defined on $x = -1$ we have that*

$$\begin{aligned}\mathbb{R} \ni \mathrm{P}(-1) &= \lim_{x \to -1^+} \mathrm{P}(x) \\ &= \lim_{x \to -1^+} f\left(\frac{x^2}{x+1}\right) \\ &= \lim_{x \to \infty} f\left(\frac{x^2}{x+1}\right) \\ &= \lim_{x \to \infty} \mathrm{P}(x)\end{aligned}$$

Since P tends to a finite value as $x \to \infty$ it must be a constant polynomial. In particular, f must be constant in the range of $\frac{x^2}{x+1}$ which is an infinite set, implying that f must also be constant. This proves what we wanted.

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3196`.

PART 5

JoM ... proposes

1 (a) Using Riemann sums evaluate the limit

$$\ell = \lim_{n\to+\infty}\left(\frac{1}{n}\ln n! - \ln n\right)$$

(b) Using the above result prove that

$$\lim_{n\to+\infty}\left(\frac{n!e^n}{n^{n+1/2}\sqrt{2\pi}}\right)^{1/n} = 1$$

(c) Prove that

$$\lim_{n\to+\infty}\frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$$

2 Let $\mathcal{H}_n$ denote the n- th harmonic number. Evaluate the limit

$$\ell = \lim_{n\to+\infty}\left(\frac{1}{\ln n}\sum_{k=1}^{n}\frac{\mathcal{H}_k}{k} - \frac{\ln n}{2}\right)$$

3 Let $f : [M, N] \to \mathbb{R}$ be a monotonic function. Prove that:

$$\left|\sum_{k=M}^{N} f(k) - \int_M^N f(x)\,\mathrm{d}x\right| \leq \max\{|f(M)|, |f(N)|\}$$

4 Let Γ denote the Euler's Gamma function. Prove that

$$\prod_{n=1}^{\infty}\sqrt[2^n]{\frac{\Gamma(2^n+\frac{1}{2})}{\Gamma(2^n)}} = \frac{8\sqrt{\pi}}{e^2}$$

5 Let $z_1, z_2, \ldots, z_n \in \mathbb{C}$ be the roots of the polynomial

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbb{C}[x]$$

Prove that:

$$\prod_{k=1}^{n}\max\{1, |z_k|\} \leq \sqrt{1 + |a_{n-1}|^2 + \cdots + |a_0|^2}$$

6 Let $\sum\limits_{n=1}^{\infty} a_n$ be a convergent series of positive terms. Prove that there exists a strictly increasing sequence b_n which is also unbounded such that the series $\sum\limits_{n=1}^{\infty} a_n b_n$ also converges.

7 Prove that

$$\sum_{n_1=1}^{n-1}\sum_{n_2=1}^{n_1-1}\sum_{n_3=1}^{n_2-1}\cdots\sum_{n_m=1}^{n_{m-1}-1} 1 = \binom{n-1}{m}$$

8 Let X, Y be topological spaces. Let $f : X \times Y \to \mathbb{R}$ be a bounded and continuous function. Is the function $g(x) = \inf_{y \in Y} f(x, y)$ continuous? Give a brief explanation.

9 Let $n \in \mathbb{N} \mid n > 1$. Prove that

$$\int_0^{\infty} \frac{n^2 x^n \ln x}{1 + x^{2n}} \, \mathrm{d}x = \frac{\pi^2}{4} \frac{\sin \frac{\pi}{2n}}{\cos^2 \frac{\pi}{2n}}$$

10 Examine the convergence of the series

$$\sum_{n=1}^{\infty} \left(1 - 2 \exp \left(\sum_{k=1}^{n} \frac{(-1)^k}{k} \right) \right)$$

11 Let $\gamma_n = \mathcal{H}_n - \ln n$. Evaluate the limit

$$\ell = \lim_{n \to +\infty} (\sin \gamma_n - \sin \gamma) \sqrt[n]{n!}$$

12 Let ϑ_3 denote one of Jacobi's theta functions and $\mathscr{n}'$ a function such that

$$\sum_{d \mid n} \mathscr{n}'(d) = \begin{cases} n & , \quad \text{if n is a perfect square} \\ 0 & , \quad \text{otherwise} \end{cases}$$

Prove that

$$\prod_{n=1}^{\infty} \left(1 - \frac{1}{\phi^n} \right)^{\frac{\mu(n) - \varphi(n) + \mathscr{n}(n)}{n}} = \sqrt{\frac{e^3}{\exp\left(\vartheta_3 \left(0; \frac{1}{\phi} \right) \right)}}$$

13 Let μ denote the Möbius function. Prove that

$$\lim_{N \to +\infty} \sum_{n=1}^{N} \frac{|\mu(n)|}{n} = +\infty$$

14 For the values of a, x for which the following series makes sense, prove that

$$\sum_{n=-\infty}^{\infty} \frac{\cos nx}{\Gamma(a + n + 1) \Gamma(a - n + 1)} = \frac{\left(2 \cos \frac{x}{2} \right)^{2a}}{\Gamma(2a + 1)}$$

15 Evaluate the integral

$$\mathcal{J} = \int \frac{\mathrm{d}x}{\sin x \sin(x + 1)}$$

6

PART

JoM ... Study

Author: Tolaso

JoM ... studies Coxeter - Abel -Ahmed integrals

Theorem • Coxeter Integral

It holds that:

$$\int_0^{\pi/4} \arctan \sqrt{\frac{\cos 2\vartheta}{2\cos^2\vartheta}}\, \mathrm{d}\vartheta = \frac{\pi^2}{24}$$

Proof. We begin by stating 3 lemmata:

Lemma 1

It holds that $\arctan x = \displaystyle\int_0^1 \frac{x}{1+x^2t^2}\,\mathrm{d}t$.

Proof. It follows immediately from the fact that $\arctan x = \displaystyle\int_0^x \frac{\mathrm{d}t}{1+t^2}$. ♦

Lemma 2

It holds that: $\displaystyle\int_0^1 \frac{\mathrm{d}x}{(x^2+1)\sqrt{x^2+2}} = \frac{\pi}{6}$.

Proof. We have successively:

$$\begin{aligned}
\int_0^1 \frac{\mathrm{d}x}{\sqrt{x^2+2}(x^2+1)} &\overset{x=\sqrt{2}\sinh t}{=\!=\!=} \int_0^{\operatorname{arcsinh}\frac{1}{\sqrt{2}}} \frac{\mathrm{d}t}{1+2\sinh^2 t} \\
&= \int_0^{\operatorname{arcsinh}\frac{1}{\sqrt{2}}} \frac{\mathrm{d}t}{\cosh 2t} \\
&= \int_0^{\operatorname{arcsinh}\frac{1}{\sqrt{2}}} \frac{\cosh 2t}{1+\sinh^2 2t}\,\mathrm{d}t \\
&= \frac{1}{2}\operatorname{arctanh}\left(\sinh 2\left(\operatorname{arcsinh}\frac{1}{2}\right)\right) \\
&= \frac{1}{2}\operatorname{arctanh}\sqrt{3} = \frac{\pi}{6}
\end{aligned}$$

♦

Lemma 3

It holds that $\displaystyle\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)} \qquad a, b \neq 0.$

Proof. We have successively:

$$\begin{aligned}\int_0^\infty \frac{\mathrm{d}x}{(x^2+a^2)(x^2+b^2)} &= \int_0^\infty \frac{1}{b^2-a^2}\left(\frac{1}{x^2+a^2}-\frac{1}{x^2+b^2}\right)\mathrm{d}x\\ &= \frac{1}{b^2-a^2}\left(\frac{\pi}{2a}-\frac{\pi}{2b}\right)\\ &= \frac{\pi}{2ab(a+b)}\end{aligned}$$

♦

We are now in position to evaluate the initial integral. We have successively:

$$\begin{aligned}\int_0^{\pi/4} \arctan\sqrt{\frac{\cos 2\vartheta}{2\cos^2\vartheta}}\,\mathrm{d}\vartheta &= \int_0^{\pi/4}\int_0^1 \frac{\sqrt{\frac{\cos 2\vartheta}{2\cos^2\vartheta}}}{1+\left(\frac{\cos 2\vartheta}{2\cos^2\vartheta}\right)x^2}\,\mathrm{d}x\,\mathrm{d}\vartheta\\ &= \int_0^1\int_0^{\pi/4} \frac{\sqrt{2}\cos\vartheta\sqrt{1-2\sin^2\vartheta}}{2-2\sin^2\vartheta+(1-2\sin^2\vartheta)x^2}\,\mathrm{d}\vartheta\,\mathrm{d}x\\ &= \int_0^1\int_0^{\pi/4} \frac{\cos\varphi\sqrt{1-\sin^2\varphi}}{2-\sin^2\varphi+\left(1-\sin^2\varphi)x^2\right)}\,\mathrm{d}\varphi\,\mathrm{d}x\\ &= \int_0^1\int_0^{\pi/4} \frac{\cos^2\varphi}{\sin^2\varphi+(x^2+2)\cos^2\varphi}\,\mathrm{d}\varphi\,\mathrm{d}x\\ &= \int_0^1\int_0^{\pi/2} \frac{\mathrm{d}\varphi\,\mathrm{d}x}{\tan^2\varphi+x^2+2}\\ &\overset{y=\tan\varphi}{=\!=} \int_0^1\int_0^\infty \frac{\mathrm{d}y\,\mathrm{d}x}{(y^2+x^2+2)(y^2+1)}\\ &= \frac{\pi}{2}\int_0^1 \frac{\mathrm{d}y}{(1+\sqrt{2+y^2})\sqrt{2+y^2}}\\ &= \frac{\pi}{2}\left(\frac{\pi}{4}-\frac{\pi}{6}\right)=\frac{\pi^2}{24}\end{aligned}$$

♦

Theorem • Ahmed Integral

It holds that:

$$\int_0^1 \frac{\arctan\sqrt{x^2+2}}{\sqrt{x^2+2}}\frac{\mathrm{d}x}{x^2+1} = \frac{5\pi^2}{96}$$

Proof. We consider the function $f(t) = \displaystyle\int_0^1 \frac{\arctan\left(t\sqrt{2+x^2}\right)}{(1+x^2)\sqrt{2+x^2}}\,\mathrm{d}x$. Differentiating f with respect to t we have:

$$\begin{aligned}
f'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{\arctan\left(t\sqrt{2+x^2}\right)}{(1+x^2)\sqrt{2+x^2}}\,\mathrm{d}x \\
&= \int_0^1 \frac{\partial}{\partial t}\frac{\arctan\left(t\sqrt{2+x^2}\right)}{(1+x^2)\sqrt{2+x^2}}\,\mathrm{d}x \\
&= \int_0^1 \frac{\mathrm{d}x}{(1+x^2)(1+2t^2+t^2x^2)} \\
&= \int_0^1 \frac{\mathrm{d}x}{(1+t^2)(1+x^2)} - \int_0^1 \frac{t^2}{(1+t^2)(1+2t^2+t^2x^2)}\,\mathrm{d}x \\
&= \frac{\pi}{4}\cdot\frac{1}{1+t^2} - \frac{t}{(1+t^2)\sqrt{1+2t^2}}\arctan\frac{t}{\sqrt{1+2t^2}}
\end{aligned}$$

Integrating the last equation from 1 to infinity we have:

$$\begin{aligned}
\int_1^\infty f'(t)\,\mathrm{d}t &= \int_1^\infty \left(\frac{\pi}{4}\cdot\frac{1}{1+t^2} - \frac{t}{(1+t^2)\sqrt{1+2t^2}}\arctan\frac{t}{\sqrt{1+2t^2}}\right)\mathrm{d}t \\
&= \frac{\pi}{4}\left(\frac{\pi}{2}-\frac{\pi}{4}\right) - \int_1^\infty \frac{t}{(1+t^2)\sqrt{1+2t^2}}\arctan\frac{t}{\sqrt{1+2t^2}}\,\mathrm{d}t \\
&= \frac{\pi^2}{16} - \int_1^\infty \frac{t}{(1+t^2)\sqrt{1+2t^2}}\arctan\frac{t}{\sqrt{1+2t^2}}\,\mathrm{d}t
\end{aligned}$$

However,

$$\begin{aligned}
\lim_{t\to+\infty} f(t) &= \lim_{t\to+\infty}\int_0^1 \frac{\arctan\left(t\sqrt{2+x^2}\right)}{(1+x^2)\sqrt{2+x^2}}\,\mathrm{d}x \\
&= \int_0^1 \lim_{t\to+\infty}\frac{\arctan\left(t\sqrt{2+x^2}\right)}{(1+x^2)\sqrt{2+x^2}}\,\mathrm{d}x \\
&= \frac{\pi}{2}\int_0^1 \frac{\mathrm{d}x}{(1+x^2)\sqrt{2+x^2}} \\
&\overset{\text{Lemma 2}}{=\!=} \frac{\pi}{2}\cdot\frac{\pi}{6} \\
&= \frac{\pi^2}{12}
\end{aligned}$$

Hence, the above equation gives us

$$\frac{\pi^2}{12} - \mathbf{f}(1) = \frac{\pi^2}{16} - \int_1^\infty \frac{\mathbf{t}}{(1+\mathbf{t}^2)\sqrt{1+2t^2}} \arctan \frac{\mathbf{t}}{\sqrt{1+2t^2}}\, \mathrm{d}\mathbf{t} \tag{1}$$

Suffice to compute the integral. Applying the change of variables $t \mapsto 1/t$ we have that:

$$\begin{aligned}
\mathcal{J} &= \int_1^\infty \frac{t}{(1+t^2)\sqrt{1+2t^2}} \arctan \frac{t}{\sqrt{1+2t^2}}\, \mathrm{d}t \\
&= \int_1^\infty \frac{1}{1+\left(\frac{1}{t}\right)^2 \sqrt{2+\frac{1}{t^2}}} \arctan \frac{1}{\sqrt{2+\frac{1}{t^2}}} \frac{\mathrm{d}t}{t^2} \\
&\overset{t\mapsto 1/t}{=\!=} \int_0^1 \frac{1}{(1+t^2)\sqrt{2+t^2}} \arctan \frac{1}{\sqrt{2+t^2}}\, \mathrm{d}t \\
&= \int_0^1 \frac{1}{(1+t^2)\sqrt{2+t^2}} \left(\frac{\pi}{2} - \arctan \sqrt{2+t^2}\right) \mathrm{d}t \\
&= \frac{\pi}{2}\int_0^1 \frac{\mathrm{d}t}{(1+t^2)\sqrt{2+t^2}} - f(1) \\
&= \frac{\pi}{2}\cdot\frac{\pi}{6} - f(1) \\
&= \frac{\pi^2}{12} - f(1)
\end{aligned}$$

Going back at (1) we have:

$$\begin{aligned}
\frac{\pi^2}{12} - f(1) = \frac{\pi^2}{16} - \left(\frac{\pi^2}{12} - f(1)\right) &\Leftrightarrow \frac{\pi^2}{12} - f(1) = \frac{\pi^2}{16} - \frac{\pi^2}{12} + f(1) \\
&\Leftrightarrow 2f(1) = \frac{5\pi^2}{48} \Leftrightarrow \boxed{\mathbf{f}(1) = \frac{5\pi^2}{96}}
\end{aligned}$$

♦

Proof. A second proof could be as follows. We begin by the well known equation:

$$\arctan x + \arctan \frac{1}{x} = \frac{\pi}{2} \qquad\qquad x > 0$$

Let us denote the given integral as $\mathcal{J}$.

$$\begin{aligned}
\mathcal{J} &= \int_0^1 \frac{\arctan \sqrt{2+x^2}}{(1+x^2)\sqrt{2+x^2}}\, \mathrm{d}x \\
&= \frac{\pi}{2}\int_0^1 \frac{\mathrm{d}x}{(1+x^2)\sqrt{2+x^2}} - \int_0^1 \frac{\arctan\left(\frac{1}{\sqrt{2+x^2}}\right)}{(1+x^2)\sqrt{2+x^2}}\, \mathrm{d}x \\
&= \frac{\pi^2}{12} - \int_0^1 \frac{\arctan\left(\frac{1}{\sqrt{2+x^2}}\right)}{(1+x^2)\sqrt{2+x^2}}\, \mathrm{d}x
\end{aligned}$$

Using the fact that

$$\arctan \frac{1}{a} = \int_0^1 \frac{a}{x^2+a^2}\, \mathrm{d}x \Leftrightarrow \frac{1}{a}\arctan\frac{1}{a} = \int_0^1 \frac{\mathrm{d}x}{x^2+a^2}$$

we get

$$\begin{aligned}
\int_0^1 \frac{\arctan\left(\frac{1}{\sqrt{2+x^2}}\right)}{(1+x^2)\sqrt{2+x^2}}\,\mathrm{d}x &= \int_0^1\int_0^1 \frac{\mathrm{d}(x,y)}{(1+x^2)(2+x^2+y^2)} \\
&= \int_0^1\int_0^1 \frac{1}{y^2+1}\left(\frac{1}{1+x^2} - \frac{1}{2+x^2+y^2}\right)\mathrm{d}(x,y) \\
&= \int_0^1\int_0^1 \frac{\mathrm{d}(x,y)}{(1+x^2)(1+y^2)} - \int_0^1\int_0^1 \frac{\mathrm{d}(x,y)}{2+x^2+y^2} \\
&= \frac{1}{2}\int_0^1\int_0^1 \frac{\mathrm{d}(x,y)}{(1+x^2)(1+y^2)} \\
&= \frac{1}{2}\left(\int_0^1 \frac{\mathrm{d}x}{1+x^2}\right)^2 \\
&= \frac{\pi^2}{32}
\end{aligned}$$

The result now follows. ♦

Theorem • Abel Integral

It holds that

$$\int_0^\infty \frac{t}{\left(e^{\pi t}-e^{-\pi t}\right)\left(1+t^2\right)}\,\mathrm{d}t = \frac{\ln 2}{2}-\frac{1}{4}$$

Proof. This result follows from Abel - Plana. We choose $f(t)=\frac{1}{2(1+t)}$ and we have:

$$i\int_0^\infty \frac{f(it)-f(-it)}{2\sinh \pi t}\,\mathrm{d}t+\frac{f(0)}{2}=\sum_{n=0}^{\infty}(-1)^n f(n) \Leftrightarrow i\int_0^\infty \frac{\frac{1}{2(1+it)}-\frac{1}{2(1-it)}}{2\sinh \pi t}\,\mathrm{d}t+\frac{1}{4}=$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{2(1+n)}$$

$$\Leftrightarrow -i^2\int_0^\infty \frac{t}{2\sinh \pi t\left(t^2+1\right)}\,\mathrm{d}t+$$

$$+\frac{1}{4}=\sum_{n=0}^{\infty}\frac{(-1)^n}{2(1+n)}$$

$$\Leftrightarrow \int_0^\infty \frac{t}{2\sinh \pi t\left(t^2+1\right)}\,\mathrm{d}t$$

$$+\frac{1}{4}=\frac{\ln 2}{2}$$

$$\Leftrightarrow \int_0^\infty \frac{t}{2\sinh \pi t\left(t^2+1\right)}\,\mathrm{d}t$$

$$=\frac{\ln 2}{2}-\frac{1}{4}$$

$$\Leftrightarrow \int_0^\infty \frac{t}{\left(e^{\pi t}-e^{-\pi t}\right)\left(1+t^2\right)}\,\mathrm{d}t$$

$$=\frac{\ln 2}{2}-\frac{1}{4}$$

♦

Applications

1 Prove that:

$$\int_0^1 \frac{1}{1+x^2} \arctan \sqrt{\frac{1-x^2}{2}}\, \mathrm{d}x = \frac{\pi^2}{24}$$

Proof. Let us denote the given integral as $\mathcal{J}$. Following the same technique we applied in the Coxeter integral we have:

$$\begin{aligned}
\mathcal{J} &= \int_0^1 \frac{1}{1+x^2} \arctan \sqrt{\frac{1-x^2}{2}}\, \mathrm{d}x \\
&= -\sqrt{2} \int_0^1 \frac{x \arctan x}{\sqrt{1-x^2}\,(3-x^2)}\, \mathrm{d}x \\
&= -\sqrt{2} \int_0^1 \frac{x}{\sqrt{1-x^2}\,(3-x^2)} \int_0^1 \frac{x}{1+x^2t^2}\, \mathrm{d}t\, \mathrm{d}x \\
&= -\sqrt{2} \int_0^1 \int_0^1 \frac{x^2}{\sqrt{1-x^2}\,(3-x^2)\left(x^2+\frac{1}{t^2}\right)} \frac{1}{t^2}\, \mathrm{d}x\, \mathrm{d}t \\
&\overset{x=\cos\partial}{=\!=} \sqrt{2} \int_0^1 \int_0^{\pi/2} \frac{\cos^2\partial}{(3-\cos^2\partial)\left(\cos^2\partial+\frac{1}{t^2}\right)}\, \mathrm{d}\partial\, \frac{\mathrm{d}t}{t^2} \\
&= \frac{\sqrt{2}}{3} \int_0^1 \int_0^{\pi/2} \frac{\sec^2\partial}{\left(\sec^2\partial-\frac{1}{3}\right)(t^2+\sec^2\partial)}\, \mathrm{d}\partial\, \mathrm{d}t \\
&= \frac{\sqrt{2}}{3} \int_0^1 \int_0^{\pi/2} \frac{\sec^2\partial}{\left(\tan^2\partial+\frac{2}{3}\right)(\tan^2\partial+1+t^2)}\, \mathrm{d}\partial\, \mathrm{d}t \\
&= \frac{\sqrt{2}}{3} \underbrace{\int_0^1 \left(\int_0^{\pi/2} \frac{\sec^2\partial}{\tan^2\partial+\frac{2}{3}}\, \mathrm{d}\partial - \int_0^{\pi/2} \frac{\sec^2\partial}{\tan^2\partial+1+t^2}\, \mathrm{d}\partial\right) \frac{\mathrm{d}t}{t^2+\frac{1}{3}}}_{\mathcal{I}}
\end{aligned}$$

For the two remaining integrals we have:

$$\begin{aligned}
\int_0^{\pi/2} \frac{\sec^2\partial}{\tan^2\partial+\frac{2}{3}}\, \mathrm{d}\partial &\overset{u=\tan\partial}{=\!=} \int_0^\infty \frac{\mathrm{d}u}{u^2+\frac{2}{3}} \\
&= \left[\frac{\sqrt{3}}{2} \arctan \sqrt{\frac{3}{2}}\,u\right]_0^\infty \\
&= \frac{\sqrt{3}\pi}{2\sqrt{2}} \\
&= \frac{\pi\sqrt{6}}{4}
\end{aligned}$$

Similarly , the other is calculated as

$$\int_0^{\pi/2} \frac{\sec^2\partial}{\tan^2\partial+1+t^2}\, \mathrm{d}\partial = \frac{\pi}{2\sqrt{1+t^2}}$$

Thus,

$$\begin{aligned}
\mathcal{I} &= \frac{\pi\sqrt{6}}{4}\int_0^1 \frac{\mathrm{d}t}{t^2+\frac{1}{3}} - \frac{\pi}{2}\int_0^1 \frac{\mathrm{d}t}{\sqrt{1+t^2}\left(t^2+\frac{1}{3}\right)} \\
&\overset{t\mapsto 1/t}{=\!=} \frac{\pi\sqrt{6}}{4}\frac{\pi}{\sqrt{3}} - \frac{3\pi}{2}\int_1^\infty \frac{t}{\sqrt{t^2+1}\,(t^2+3)}\,\mathrm{d}t \\
&\overset{t\mapsto t^2}{=\!=} \frac{\pi^2\sqrt{2}}{4} - \frac{3\pi}{4}\int_1^\infty \frac{\mathrm{d}t}{\sqrt{t+1}(t+3)} \\
&= \frac{\pi^2\sqrt{2}}{4} - \frac{3\pi}{4}\int_2^\infty \frac{\mathrm{d}t}{\sqrt{t}\,(t+2)} \\
&\overset{t\mapsto t^2}{=\!=} \frac{\pi^2\sqrt{2}}{4} - \frac{3\pi}{2}\int_{\sqrt{2}}^\infty \frac{\mathrm{d}t}{t^2+2} \\
&= \frac{\pi^2\sqrt{2}}{4} - \frac{3\pi^2}{8\sqrt{2}}
\end{aligned}$$

Collecting , we have

$$\begin{aligned}
\mathcal{J} &= \frac{\sqrt{2}}{3}\left(\frac{\pi^2\sqrt{2}}{4} - \frac{3\sqrt{2}\pi^2}{16}\right) \\
&= \frac{\sqrt{2}}{3}\cdot\frac{\sqrt{2}\pi^2}{16} \\
&= \frac{\pi^2}{24}
\end{aligned}$$

♦

2 Prove that:

$$\int_0^{\pi/4} \arctan\left(\frac{\sqrt{2}\cos 3\varphi}{(2\cos 2\varphi+3)\,\sqrt{\cos 2\varphi}}\right)\mathrm{d}\varphi = 0$$

Proof. By applying the change of variables $\varphi \mapsto \arctan t$ and integrating by parts we get the the previous integral. For example , if we denote by $\mathcal{J}$ the initial integral we have:

$$\begin{aligned}
\mathcal{J} &= \int_0^1 \frac{1}{1+t^2}\arctan\left(\frac{\sqrt{2}(1-3t^2)}{(5+t^2)\sqrt{1-t^2}}\right)\mathrm{d}t \\
&= \frac{\pi^2}{8} - \int_0^1 \frac{3\sqrt{2}\,t\arctan t}{(3-t^2)\sqrt{1-t^2}}\,\mathrm{d}t \\
&= \frac{\pi^2}{8} - 3\int_0^1 \frac{1}{1+t^2}\arctan\sqrt{\frac{1-t^2}{2}}\,\mathrm{d}t \\
&= 0
\end{aligned}$$

♦

(3) Let $n \in \mathbb{N}$. Prove that:

$$\int_0^1 \frac{\arctan^n\left(x\sqrt{2+x^2}\right)}{\sqrt{x^2+2}} \frac{2}{1+x^2}\,\mathrm{d}x = \left(\frac{\pi}{3}\right)^{n+1} \cdot \frac{1}{n+1}$$

Proof. Applying the change of variables $x \mapsto \arctan\left(x\sqrt{2+x^2}\right)$ we get:

$$\begin{aligned}\int_0^1 \frac{\arctan^n\left(x\sqrt{2+x^2}\right)}{\sqrt{x^2+2}} \frac{2}{1+x^2}\,\mathrm{d}x &= \int_0^{\arctan\sqrt{3}} u^n \mathrm{d}u \\ &= \left[\frac{u^{n+1}}{n+1}\right]_0^{\frac{\pi}{3}} \\ &= \left(\frac{\pi}{3}\right)^{n+1} \cdot \frac{1}{n+1}\end{aligned}$$

This completes the proof. ♦

(4) Prove that:

$$\int_0^1 \frac{1}{(1+x^2)\sqrt{2+x^2}} \arctan\left(\sqrt{2+x^2} \cdot \frac{x+1}{x^2-x+2}\right) \mathrm{d}x = \frac{13\pi^2}{288}$$

Proof. The key here is the equality

$$\arctan\left(\sqrt{2+x^2} \cdot \frac{x+1}{x^2-x+2}\right) = \arctan\frac{1}{\sqrt{x^2+2}} + \arctan\frac{x}{\sqrt{2+x^2}}$$

which comes naturally from the identity

$$\arctan x + \arctan y = \arctan\frac{x+y}{1-xy}$$

Hence , we have to calculate the following integrals.

$$\int_0^1 \frac{1}{(1+x^2)\sqrt{2+x^2}} \arctan\frac{1}{\sqrt{2+x^2}}\,\mathrm{d}x$$

$$\int_0^1 \frac{1}{(1+x^2)\sqrt{2+x^2}} \arctan\frac{x}{\sqrt{2+x^2}}\,\mathrm{d}x$$

For the first integral we have:

$$\begin{aligned}\int_0^1 \frac{1}{(1+x^2)\sqrt{2+x^2}} \arctan\frac{1}{\sqrt{2+x^2}}\,\mathrm{d}x &= \int_0^1 \frac{\frac{\pi}{2} - \arctan\sqrt{2+x^2}}{(1+x^2)\sqrt{2+x^2}}\,\mathrm{d}x \\ &= \frac{\pi}{2}\cdot\frac{\pi}{6} - \frac{5\pi^2}{96} \\ &= \frac{\pi^2}{32}\end{aligned}$$

from Lemma 2 and the Ahmed integral. For the second integral we have:

$$\begin{aligned}\int_0^1 \frac{1}{(1+x^2)\sqrt{2+x^2}} \arctan \frac{x}{\sqrt{2+x^2}}\,\mathrm{d}x &\overset{x\mapsto 1/x}{=\!=} \int_1^\infty \frac{x \arctan \frac{1}{\sqrt{1+2x^2}}}{(1+x^2)\sqrt{1+2x^2}}\,\mathrm{d}x \\ &\overset{x\mapsto x^2}{=\!=} \frac{1}{2}\int_1^\infty \frac{\arctan \frac{1}{\sqrt{1+2x}}}{(1+x)\sqrt{1+2x}}\,\mathrm{d}x \\ &= \left[-\frac{1}{2}\arctan^2 \frac{1}{\sqrt{1+2x}}\right]_1^\infty \\ &= \frac{\pi^2}{72}\end{aligned}$$

Hence,

$$\int_0^1 \frac{1}{(1+x^2)\sqrt{2+x^2}} \arctan\left(\sqrt{2+x^2}\cdot\frac{x+1}{x^2-x+2}\right)\mathrm{d}x = \frac{\pi^2}{32}+\frac{\pi^2}{72} = \frac{13\pi^2}{288}$$

♦

5 Let $a > 1$. Prove that:

$$\int_0^\infty \frac{\arctan\sqrt{a^2+x^2}}{(x^2+1)\sqrt{a^2+x^2}}\,\mathrm{d}x = \frac{\pi\left(2\arctan\sqrt{a^2-1} - \arctan\sqrt{a^4-1}\right)}{2\sqrt{a^2-1}}$$

Proof. We are applying the same technique we used in the Coxeter integral. Writing

$$\arctan\sqrt{a^2+x^2} = \int_0^1 \frac{\sqrt{a^2+x^2}}{1+(a^2+x^2)\,y^2}\,\mathrm{d}y$$

we have successively

$$\begin{aligned}\int_0^\infty \frac{\arctan\left(\sqrt{a^2+x^2}\right)}{(1+x^2)\sqrt{a^2+x^2}}\,\mathrm{d}x &= \int_0^\infty\int_0^1 \frac{\mathrm{d}y}{1+y^2(a^2+x^2)}\,\frac{\mathrm{d}x}{1+x^2} \\ &= \int_0^1\int_0^\infty \frac{\mathrm{d}x}{(1+x^2)(1+y^2(a^2+x^2))}\,\mathrm{d}y \\ &= \int_0^1 \frac{1}{1+(a^2-1)y^2}\int_0^\infty\left(\frac{1}{1+x^2} - \frac{1}{a^2+\frac{1}{y^2}+x^2}\right)\mathrm{d}x\,\mathrm{d}y \\ &= \frac{\pi}{2}\int_0^1 \frac{\mathrm{d}y}{\sqrt{1+a^2y^2}(y+\sqrt{1+a^2y^2})} \\ &\overset{y=\frac{2t}{a(1-t^2)}}{=\!=} \pi\int_0^{\frac{a}{1+\sqrt{1+a^2}}} \frac{\mathrm{d}t}{a(1+t^2)+2t} \\ &= \frac{\pi a}{a^2-1}\int_0^{\frac{a}{1+\sqrt{1+a^2}}} \frac{\mathrm{d}t}{1+\left(\frac{1+at}{\sqrt{a^2-1}}\right)^2} \\ &\overset{\frac{1+at}{\sqrt{a^2-1}}=\frac{1}{u}}{=\!=} \frac{\pi}{\sqrt{a^2-1}}\int_{\sqrt{\frac{a^2-1}{a^2+1}}}^{\sqrt{a^2-1}} \frac{\mathrm{d}u}{1+u^2}\end{aligned}$$

$$= \frac{\pi}{\sqrt{a^2-1}}\left[\arctan\sqrt{a^2-1} - \arctan\sqrt{\frac{a^2-1}{a^2+1}}\right]$$

$$= \frac{\pi}{\sqrt{a^2-1}}\left[\arctan\sqrt{a^2-1} - \frac{1}{2}\arctan\sqrt{a^4-1}\right]$$

and this completes the proof. We can also slightly improve the result

$$\int_0^\infty \frac{\arctan\left(\sqrt{a^2+x^2}\right)}{(1+x^2)\sqrt{a^2+x^2}}\,\mathrm{d}x = \frac{\pi}{\sqrt{a^2-1}}\arctan\frac{\sqrt{a^2-1}}{2+\sqrt{a^2+1}}$$

♦

Suggested Problems

1. Prove the following:

(a) $$\int_0^{\pi/2} \arccos\left(\frac{\cos x}{2\cos x + 1}\right) \mathrm{d}x = \frac{5\pi^2}{24}.$$

(b) $$\int_0^{\pi/3} \arccos\left(\frac{\cos x}{2\cos x + 1}\right) \mathrm{d}x = \frac{2\pi^2}{15}.$$

(c) $$\int_0^{\pi/5} \arccos\left(\frac{\cos x}{2\cos x + 1}\right) \mathrm{d}x = \frac{71\pi^2}{900}.$$

2. Prove that:

$$\int_0^{\sqrt{2}/2} \frac{\arcsin x^2}{\sqrt{1 + x^2}(1 + 2x^2)} \mathrm{d}x = \frac{\pi^2}{144}$$

3. Prove that:

$$\int_0^{\pi/3} \arccos\left(\frac{1 - \cos \partial}{2\cos \partial}\right) \mathrm{d}\partial = \frac{11\pi^2}{72}$$

4. Prove that:

$$\int_0^1 \frac{\arctan \sqrt{x^2 + 2}}{\sqrt{x^2 + 2}\,(5x^4 + 10x^2 + 1)} \mathrm{d}x = \frac{37\pi^2}{1440}$$

5. Let $a \neq 0$. Prove that:

$$\int_0^1 \frac{2a^2}{a^2 + x^2} \frac{\arctan \sqrt{2a^2 + x^2}}{\sqrt{2a^2 + x^2}} \mathrm{d}x = \pi \arctan \frac{1}{\sqrt{2a^2 + 1}} - \left(\arctan \frac{1}{a}\right)^2$$

Author: Tolaso

JoM ... studies Poisson Integrals

1 Let $a \geq 0$. We will prove that

$$I(a) = \int_0^{\pi} \ln\left(1 - 2a\cos x + a^2\right) \, \mathrm{d}x = \begin{cases} 0 & , \quad |a| \leq 1 \\ 2\pi \ln|a| & , \quad \text{otherwise} \end{cases}$$

Proof. We present a simple yet powerful method to tackle this particular integral. For that reason we break the solution into two main steps.

(a) Making the substitution $x \mapsto \pi - x$ we note that

$$I(a) = I(-a) \tag{1}$$

(b) We also note that

$$\begin{aligned} I(a) + I(-a) &= \int_0^{\pi} \ln\Big(\left(1 - 2a\cos x + a^2\right)\left(1 + 2a\cos x + a^2\right)\Big) \, \mathrm{d}x \\ &= \int_0^{\pi} \ln\Big(\left(1 + a^2\right)^2 - (2a\cos x)^2\Big) \, \mathrm{d}x \end{aligned}$$

Using double angle formulae we get

$$\begin{aligned} I(a) + I(-a) &= \int_0^{\pi} \ln\left(1 + 2a^2 + a^4 - 2a^2(1 + \cos 2x)\right) \, \mathrm{d}x \\ &= \int_0^{\pi} \ln\left(1 - 2a^2\cos 2x + a^4\right) \, \mathrm{d}x \end{aligned}$$

so by setting $x \mapsto \frac{x}{2}$ we get

$$I(a) + I(-a) = \frac{1}{2} \int_0^{2\pi} \ln\left(1 - 2a^2\cos x + a^4\right) \, \mathrm{d}x \tag{2}$$

Splitting the integral at π and setting $x \mapsto 2\pi - x$ at the second integral we get

$$\begin{aligned} I(a) + I(-a) &= \frac{1}{2} I\left(a^2\right) + \frac{1}{2} \int_{\pi}^{2\pi} \ln\left(1 - 2a^2\cos x + a^4\right) \, \mathrm{d}x \\ &= \frac{1}{2} I\left(a^2\right) + \frac{1}{2} \int_0^{\pi} \ln\left(1 - 2a^2\cos x + a^4\right) \, \mathrm{d}x \\ &= I(a^2) \end{aligned}$$

Hence

$$I(a) = \frac{1}{2} I(a^2) \tag{3}$$

It follows from (3) that $I(0) = 0$ and $I(1) = 0$.We now distinguish cases:

(i) $0 \le a < 1$. Iterating n times (3) we get:

$$I(a) = \frac{1}{2^n} I\left(a^{2^n}\right)$$

Letting $n \to +\infty$ we get that $I(a) = 0$.

(ii) When $a > 1$ it follows that $0 < \frac{1}{a} < 1$ and consequently $I\left(\frac{1}{a}\right) = 0$. Thus,

$$\begin{aligned} I(a) &= \int_0^\pi \ln\left(a^2\left(\frac{1}{a^2} + \frac{\cos x}{a} + 1\right)\right) \, dx \\ &= 2\pi \ln a + I\left(\frac{1}{a}\right) \\ &= 2\pi \ln a \end{aligned}$$

Extending the result for negative a using (1) we get to our conclusion.

♦

2 For $|a| < 1$ evaluate the integral

$$\int_0^\pi \frac{d\partial}{1 - 2a\cos\partial + a^2}$$

Proof. For $|a| < 1$ it holds

$$1 + 2\sum_{n=1}^{\infty} a_n \cos n\partial = \frac{1 - a^2}{1 - 2a\cos\partial + a^2} \quad \Rightarrow \tag{1}$$

$$\frac{1}{1-a^2} + \frac{2}{1-a^2}\sum_{n=1}^{\infty} a_n \cos n\partial = \frac{1}{1 - 2a\cos\partial + a^2} \tag{2}$$

Integrating (2) we get that:

$$\begin{aligned} \int_0^\pi \frac{d\partial}{1 - 2a\cos\partial + a^2} &= \frac{1}{1-a^2}\int_0^\pi d\partial + \frac{2}{1-a^2}\int_0^\pi \sum_{n=1}^{\infty} a^n \cos n\partial \, d\partial \\ &= \frac{\pi}{1-a^2} + \frac{2}{1-a^2}\sum_{n=1}^{\infty} a^n \int_0^\pi \cos n\partial \, d\partial \\ &= \frac{\pi}{1-a^2} + \frac{2}{1-a^2}\sum_{n=1}^{\infty} a_n \cdot \frac{\overbrace{\sin n\pi}^{0}}{n} \\ &= \frac{\pi}{1-a^2} \end{aligned}$$

♦

Proof. One more way of calculating the integral

$$\mathcal{J} = \int_0^\pi \frac{1 - a^2}{1 - 2a\cos\partial + a^2} \, d\partial$$

is the following. It is pretty quite known from the theory of Fourier Analysis that:

$$\frac{1-a^2}{1-2a\cos\vartheta+a^2}=\sum_{n=-\infty}^{\infty}a^{|n|}e^{in\vartheta}$$

In the unit disk this series converges uniformly and therefore the interchange of integral and summation is allowed. Also, the Poisson kernel (denoted as $P_a(\vartheta)$) satisfies the following property (among other interesting ones too)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}P_a(\vartheta)\,\mathrm{d}\vartheta=1 \tag{3}$$

Hence:

$$\begin{aligned}\int_0^{\pi}\frac{1-a^2}{1-2a\cos\vartheta+a^2}\,\mathrm{d}\vartheta &= \frac{1}{2}\int_{-\pi}^{\pi}\frac{1-a^2}{1-2a\cos\vartheta+a^2}\,\mathrm{d}\vartheta\\ &= \frac{1}{2}\int_{-\pi}^{\pi}\sum_{n=-\infty}^{\infty}a^{|n|}e^{in\vartheta}\,\mathrm{d}\vartheta\\ &\overset{(3)}{=}\frac{2\pi}{2}\\ &=\pi\end{aligned}$$

♦

3 Let $0\leq a,b\leq\pi$ and $k>0$. Prove that

$$\int_0^{\infty}\frac{1}{x}\ln\left(\frac{x^2+2kx\cos b+k^2}{x^2+2kx\cos a+k^2}\right)\mathrm{d}x=a^2-b^2$$

Proof. We state two lemmata:

Lemma 1

Let $|x|<1$. It holds that

$$\sum_{n=1}^{\infty}\frac{x^n\cos(na)}{n}=-\frac{1}{2}\ln\left(x^2-2x\cos a+1\right)$$

Proof. It follows immediately from the MacLaurin expansion of the $\ln(1-z)$ by setting $z=xe^{ia}$. ♦

Lemma 2

Let $|x|<\pi$. It holds that

$$\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}=\frac{x^2}{4}-\frac{\pi^2}{12}$$

Proof. It follows by integrating the well known series

$$\sum_{n=1}^{\infty}(-1)^n\frac{\sin nx}{n}=-\frac{x}{2}\quad,\quad |x|<\pi$$

♦

To finish things off , we have successively:

$$\begin{aligned}
\int_0^{\infty}\frac{1}{x}\ln\left(\frac{x^2+2kx\cos b+k^2}{x^2+2kx\cos a+k^2}\right)\mathrm{d}x &\overset{u=x/k}{=\!=\!=} \int_0^{\infty}\frac{1}{u}\ln\left(\frac{u^2+2u\cos b+1}{u^2+2u\cos a+1}\right)\mathrm{d}u\\
&=\int_0^{1}\frac{1}{u}\ln\left(\frac{u^2+2u\cos b+1}{u^2+2u\cos a+1}\right)\mathrm{d}u+\\
&\quad+\int_1^{\infty}\frac{1}{u}\ln\left(\frac{u^2+2u\cos b+1}{u^2+2u\cos a+1}\right)\mathrm{d}u\\
&\overset{u\mapsto 1/u}{=\!=\!=}2\int_0^{1}\frac{1}{u}\ln\left(\frac{u^2+2u\cos b+1}{u^2+2u\cos a+1}\right)\mathrm{d}u\\
&=4\int_0^{1}\frac{1}{u}\sum_{n=1}^{\infty}(-u)^n\frac{\cos na-\cos nb}{n}\,\mathrm{d}u\\
&=4\sum_{n=1}^{\infty}(-1)^n\frac{\cos na-\cos nb}{n}\int_0^1 u^{n-1}\,\mathrm{d}u\\
&=4\sum_{n=1}^{\infty}(-1)^n\frac{\cos na-\cos nb}{n^2}\\
&=4\left(\frac{a^2}{4}-\frac{b^2}{4}\right)\\
&=a^2-b^2
\end{aligned}$$

♦

Proof. Another proof is by invoking a double integral by noting that

$$\ln\left(\frac{x^2+2kx\cos b+k^2}{x^2+2kx\cos a+k^2}\right)=\ln\left(x^2+2kx\cos a+k^2\right)\Big|_{a=a}^{a=b}$$

Hence,

$$\begin{aligned}
\int_0^{\infty}\frac{1}{x}\ln\left(\frac{x^2+2kx\cdot\cos b+k^2}{x^2+2kx\cdot\cos a+k^2}\right)\mathrm{d}x &= \int_0^{\infty}\frac{1}{x}\ln\left(x^2+2kx\cos a+k^2\right)\Big|_{a=a}^{a=b}\mathrm{d}x\\
&=\int_0^{\infty}\frac{1}{x}\int_a^b\frac{\mathrm{d}}{\mathrm{d}a}\ln\left(x^2+2kx\cos a+k^2\right)\mathrm{d}a\,\mathrm{d}x\\
&=-\int_a^b\int_0^{\infty}\frac{2k\sin a}{x^2+2k\cos ax+k^2}\,\mathrm{d}x\,\mathrm{d}a\\
&=-\int_a^b\int_0^{\infty}\frac{2k\sin a}{(x+k\cos a)^2+k^2\sin^2 a}\,\mathrm{d}x\,\mathrm{d}a\\
&=-2\int_a^b\tan^{-1}\left(\frac{x}{k\sin a}+\frac{1}{\tan a}\right)\Big|_0^{\infty}\mathrm{d}a
\end{aligned}$$

$$
\begin{aligned}
&= -2\int_a^b \left[\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{\tan a}\right)\right] \mathrm{d}a \\
&= -2\int_a^b \tan^{-1}(\tan a)\ \mathrm{d}a \\
&= -2\int_a^b a\,\mathrm{d}a \\
&= a^2 - b^2
\end{aligned}
$$

♦

(4) Let $a \in [-\pi, \pi]$. Prove that

$$\int_0^1 \frac{1}{x}\ln\left(\frac{1+2x\cos a + x^2}{1-2x\cos a + x^2}\right) \mathrm{d}x = \frac{\pi^2}{2} - \pi\,|a|$$

Proof. For $a \in [0, \pi]$ we have:

$$
\begin{aligned}
\ln\left(\frac{1+2x\cos a + x^2}{1-2x\cos a + x^2}\right) &= \ln\left(1+2x\cos a + x^2\right) - \ln\left(1-2x\cos a + x^2\right) \\
&= -2\sum_{n=1}^{\infty} \frac{(-x)^n \cos na}{n} + 2\sum_{n=1}^{\infty} \frac{x^n \cos na}{n} \\
&= -2\sum_{n=1}^{\infty} \frac{((-x)^n - x^n)\cos na}{n} \\
&= 4\sum_{n=0}^{\infty} \frac{x^{2n+1}\cos(2n+1)a}{2n+1}
\end{aligned}
$$

since $(-x)^n - x^n = 0$ when n is even. Hence,

$$
\begin{aligned}
\int_0^1 \frac{1}{x}\ln\left(\frac{1+2x\cos a + x^2}{1-2x\cos a + x^2}\right) \mathrm{d}x &= 4\int_0^1 \frac{1}{x}\sum_{n=0}^{\infty} \frac{x^{2n+1}\cos(2n+1)a}{2n+1}\,\mathrm{d}x \\
&= 4\sum_{n=0}^{\infty} \frac{\cos(2n+1)a}{2n+1}\int_0^1 x^{2n}\,\mathrm{d}x \\
&= 4\sum_{n=0}^{\infty} \frac{\cos(2n+1)a}{(2n+1)^2}
\end{aligned}
$$

The above is nothing else than the χ_2 Legendre function. Thus,

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{\cos(2n+1)a}{(2n+1)^2} &= \Re\mathfrak{e}\left(\sum_{n=0}^{\infty} \frac{e^{i(2n+1)a}}{(2n+1)^2}\right) \\
&= \frac{1}{2}\Re\mathfrak{e}\left(\mathrm{Li}_2\left(e^{ia}\right) - \mathrm{Li}_2\left(-e^{ia}\right)\right)
\end{aligned}
$$

Using Lemma 2 from the above exercise we get that

$$4\sum_{n=0}^{\infty}\frac{\cos(2n+1)a}{(2n+1)^2} = 2\left(\frac{\pi^2}{6} - \frac{\pi a}{2} + \frac{a^2}{4} - \frac{a^2}{4} + \frac{\pi^2}{12}\right)$$
$$= \frac{\pi^2}{2} - \pi a$$

Since the LHS is even we can extend the result to $a \in [-\pi, 0]$. Hence,

$$\int_0^1 \frac{1}{x}\ln\left(\frac{1+2x\cos a + x^2}{1-2x\cos a + x^2}\right)\mathrm{d}x = \frac{\pi^2}{2} - \pi\,|a|$$

♦

(5) Let $a \in \mathbb{R}$. Evaluate the integral:

$$\int_0^{\pi} \frac{2a - 2\cos x}{1 - 2a\cos x + a^2}\,\mathrm{d}x$$

Proof. Left to the reader. ♦

Author: Tolaso

JoM … studies logarithmic - trigonometric integrals

The most well known integral is

$$\int_0^{\pi/2} \ln \sin x \, dx = \int_0^{\pi/2} \ln \cos x \, dx = -\frac{\pi \ln 2}{2}$$

In this section we're generalising this integral.

Theorem 1

Let Cl_2 denote the Clausen function and let $0 \leq \partial \leq 2\pi$. It holds that:

$$\int_0^{\partial} \ln \sin x \, dx = -\frac{1}{2}\mathrm{Cl}_2(2\partial) - \partial \ln 2$$

Proof. The Fourier series of $\ln \sin x$ is given by the formula

$$\ln \sin x = -\sum_{k=1}^{\infty} \frac{\cos 2kx}{k} - \ln 2$$

Hence,

$$\begin{aligned}
\int_0^{\partial} \ln \sin x \, dx &= \int_0^{\partial} \left(-\ln 2 - \sum_{k=1}^{\infty} \frac{\cos 2kx}{k} \right) d\partial \\
&= -\partial \ln 2 - \int_0^{\partial} \sum_{k=1}^{\infty} \frac{\cos 2kx}{k} \, dx \\
&= -\partial \ln 2 - \sum_{k=1}^{\infty} \frac{1}{k} \int_0^{\partial} \cos 2kx \, dx \\
&= -\partial \ln 2 - \frac{1}{2} \sum_{k=1}^{\infty} \frac{\sin 2\partial k}{k^2} \\
&= -\partial \ln 2 - \frac{1}{2}\mathrm{Cl}_2\left(2\partial\right)
\end{aligned}$$

♦

Similarly , using the identity

$$\ln \cos x = -\ln 2 - \sum_{k=1}^{\infty} \frac{(-1)^k \cos 2kx}{k}$$

we get the following theorem

Theorem 2

Let Cl_2 denote the Clausen function and let $0 \leq \partial \leq 2\pi$. It holds that:

$$\int_0^{\partial} \ln \cos x \, dx = \frac{1}{2}\mathrm{Cl}_2(\pi - 2\partial) - \partial \ln 2$$

Proof. Left to the reader. ♦

Theorem 3

Let Ti_2 denote the Inverse Tangent Integral and let $0 \leq \partial \leq \frac{\pi}{2}$. It holds that:

$$\int_0^{\partial} \ln^n \tan x \, \mathrm{d}x = n! \sum_{k=0}^{n} (-1)^k \frac{\ln^{n-k} \tan \partial}{(n-k)!} \mathrm{Ti}_{k+1} (\tan \partial)$$

Proof. We have successively:

$$\begin{aligned}
\int_0^{\partial} \ln^m \tan x \, \mathrm{d}x &= \int_0^{\tan \partial} \frac{\ln^m x}{x^2+1} \, \mathrm{d}x \\
&= \int_0^{\tan \partial} \ln^m x \sum_{k=0}^{\infty} (-1)^k x^{2k} \, \mathrm{d}x \\
&= \sum_{k=0}^{\infty} (-1)^k \int_0^{\tan \partial} x^{2k} \ln^m x \, \mathrm{d}x \\
&\overset{x=\tan \partial y}{=\!=} \sum_{k=0}^{\infty} (-1)^k \tan^{2k+1} \partial \int_0^1 y^{2k} \left(\ln \tan \partial + \ln y \right)^m \, \mathrm{d}y \\
&= \sum_{j=0}^{m} \binom{m}{j} (\ln \tan \partial)^{m-j} \sum_{k=0}^{\infty} (-1)^k \tan^{2k+1} \partial \int_0^1 y^{2k} \ln^j y \, \mathrm{d}y \\
&= \sum_{j=0}^{m} (-1)^j j! \binom{m}{j} \ln^{m-j} \tan \partial \sum_{k=0}^{\infty} \frac{(-1)^k (\tan \partial)^{2k+1}}{(2k+1)^{j+1}} \\
&= m! \sum_{j=0}^{m} (-1)^j \frac{(\ln \tan \partial)^{m-j}}{(m-j)!} \sum_{k=0}^{\infty} \frac{(-1)^k (\tan \partial)^{2k+1}}{(2k+1)^{j+1}} \\
&= m! \sum_{j=0}^{m} (-1)^j \frac{\ln^{m-j} \tan \partial}{(m-j)!} \mathrm{Ti}_{j+1} (\tan \partial)
\end{aligned}$$

This completes the proof. ♦

Lemma 1

It holds that

$$\int_0^{\pi/2} \ln \sin x \, \mathrm{d}x = -\frac{\pi \ln 2}{2}$$

Proof. Indeed , setting $\partial = \frac{\pi}{2}$ back at theorem (1) we get:

$$\begin{aligned}
\int_0^{\pi/2} \ln \sin x \, \mathrm{d}x &= -\frac{1}{2} \mathrm{Cl}_2 \left(2 \cdot \frac{\pi}{2} \right) - \frac{\pi \ln 2}{2} \\
&= -\frac{1}{2} \sum_{n=1}^{\infty} \frac{\sin n\pi}{n^2} - \frac{\pi \ln 2}{2} \\
&= -\frac{\pi \ln 2}{2}
\end{aligned}$$

♦

Lemma 2

It holds that

$$\int_0^{\pi/4} \ln \sin x \, \mathrm{d}x = -\frac{\mathcal{G}}{2} - \frac{\pi \log 2}{4}$$

where $\mathcal{G}$ is the Catalan's constant.

Proof. Setting $x = \frac{\pi}{4}$ back at theorem (1) we have that

$$\int_0^{\pi/4} \ln \sin x \, \mathrm{d}x = -\frac{1}{2}\mathrm{Cl}_2\left(2 \cdot \frac{\pi}{4}\right) - \frac{\pi \log 2}{4}$$

On the other hand,

$$\begin{aligned}
\mathrm{Cl}_2\left(\frac{\pi}{2}\right) &= \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{2}}{n^2} \\
&= \cancelto{0}{\sum_{n=1}^{\infty} \frac{\sin \frac{2n\pi}{2}}{4n^2}} + \sum_{n=0}^{\infty} \frac{\sin\left(\frac{(2n+1)\pi}{2}\right)}{(2n+1)^2} \\
&= \sum_{n=0}^{\infty} \frac{\sin\left(\frac{(2n+1)\pi}{2}\right)}{(2n+1)^2} \\
&= \sum_{n=0}^{\infty} \frac{\sin\left(n\pi + \frac{\pi}{2}\right)}{(2n+1)^2} \\
&= \sum_{n=0}^{\infty} \frac{\cos n\pi}{(2n+1)^2} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \\
&= \mathcal{G}
\end{aligned}$$

♦

Lemma 3

It holds that

$$\int_0^{\pi/4} \ln \cos x \, \mathrm{d}x = \frac{\mathcal{G}}{2} - \frac{\pi \log 2}{4}$$

where $\mathcal{G}$ is the Catalan's constant.

Proof. Set $\partial = \frac{\pi}{4}$ at theorem (2). Proof is left to the reader. ♦

Theorem 4

Let $n \in \mathbb{N}$ and β denote the Beta Dirichlet function. It holds that:

$$\int_0^{\pi/4} \ln^n \tan x \, \mathrm{d}x = (-1)^n n! \beta(n+1)$$

Proof. We apply the change of variables $x \mapsto \tan x$, thus:

$$\begin{aligned}
\int_0^{\pi/4} \ln^n \tan x \,\mathrm{d}x &\xlongequal{x\mapsto\tan x} \int_0^1 \frac{\ln^n x}{1+x^2}\,\mathrm{d}x \\
&= \int_0^1 \ln^n x \sum_{k=0}^{\infty} (-1)^n x^{2k}\,\mathrm{d}x \\
&= \sum_{k=0}^{\infty} (-1)^n \int_0^1 x^{2k} \ln^n x \,\mathrm{d}x \\
&= (-1)^n n! \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{n+1}} \\
&= (-1)^n n! \beta(n+1)
\end{aligned}$$

♦

Lemma 4

Let $n \in \mathbb{N}$ and β denote the Beta Dirichlet function. It holds that:

$$\int_0^{\pi/2} \ln^n \tan x \,\mathrm{d}x = \begin{cases} 0 & , \quad n \text{ odd} \\ 2n!\beta(n+1) & , \quad n \text{ even} \end{cases}$$

Proof. We apply the change of variables $x \mapsto \tan x$, thus:

$$\begin{aligned}
\int_0^{\pi/2} \ln^n \tan x \,\mathrm{d}x &= \int_0^{\infty} \frac{\log^n x}{1+x^2}\,\mathrm{d}x \\
&= \int_0^1 \frac{\log^n x}{1+x^2}\,\mathrm{d}x + \int_1^{\infty} \frac{\log^n x}{1+x^2}\,\mathrm{d}x \\
&= \int_0^1 \frac{\log^n x}{1+x^2}\,\mathrm{d}x - \int_1^{0} \frac{\log^n \left(\frac{1}{x}\right)}{1+\left(\frac{1}{x}\right)^2} \frac{1}{x^2}\,\mathrm{d}x \\
&= \int_0^1 \frac{\log^n x}{1+x^2}\,\mathrm{d}x + \int_0^{1} \frac{\log^n \left(\frac{1}{x}\right)}{1+x^2}\,\mathrm{d}x \\
&= \int_0^1 \frac{\log^n x}{1+x^2}\,\mathrm{d}x + \int_0^{1} \frac{(-1)^n \log^n x}{1+x^2}\,\mathrm{d}x \\
&= (1+(-1)^n) \int_0^1 \frac{\log^n x}{1+x^2}\,\mathrm{d}x \\
&= (1+(-1)^n) \int_0^1 \log^n x \sum_{k=0}^{\infty} (-1)^k x^{2k}\,\mathrm{d}x \\
&= (1+(-1)^n) \sum_{k=0}^{\infty} (-1)^k \int_0^1 x^{2k} \log^n x \,\mathrm{d}x \\
&= (1+(-1)^n)\,(-1)^n n! \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{n+1}} \\
&= (1+(-1)^n)\,(-1)^n n! \beta(n+1)
\end{aligned}$$

The result follows by distinguishing cases for n. ♦

Theorem 5 (Duplication Formula of even order)

Let $\mathbf{Cl}_{2n}$ denote the Clausen function of even order. It holds that:

$$\mathrm{Cl}_{2m}(2\partial) = 2^{2m-1}\left[\mathrm{Cl}_{2m}(\partial) - \mathrm{Cl}_{2m}(\pi - \partial)\right]$$

Proof. Indeed, we have:

$$\begin{aligned}
\mathrm{Cl}_{2m}(\partial) - \mathrm{Cl}_{2m}(\pi - \partial) &= \sum_{n=1}^{\infty} \frac{\sin n\partial}{n^{2m}} - \sum_{n=1}^{\infty} \frac{\sin n(\pi - \partial)}{n^{2m}} \\
&= \sum_{n=1}^{\infty} \frac{\sin n\partial}{n^{2m}} - \sum_{n=1}^{\infty} \frac{\sin n\pi \cos n\partial - \cos n\pi \sin n\partial}{n^{2m}} \\
&= \sum_{n=1}^{\infty} \frac{\sin n\partial}{n^{2m}} + \sum_{n=1}^{\infty} \frac{(-1)^n \sin n\partial}{n^{2m}} \\
&= 2\sum_{n=1}^{\infty} \frac{\sin 2n\partial}{(2n)^{2m}} \\
&= \frac{1}{2^{2m-1}}\mathrm{Cl}_{2m}(2\partial)
\end{aligned}$$

♦

Theorem 6 (Duplication Formula of odd order)

Let $\mathbf{Cl}_{2n+1}$ denote the Clausen function of odd order. It holds that:

$$\mathrm{Cl}_{2m+1}(2\partial) = \zeta(2m+1) + 2^{2m}\left[\mathrm{Cl}_{2m+1}(\partial) - \mathrm{Cl}_{2m+1}(\pi - \partial)\right]$$

Proof. Integrating the duplication formula of even order we have the result. Here ζ denotes the Riemann zeta function. ♦

Suggested Problems

1. Let $m \in \mathbb{N}$. Evaluate the integral:

$$\int_0^\infty \frac{\ln x}{(1+x^2)^m}\,\mathrm{d}x$$

2. Let $\mathcal{H}_n$ denote the n - th harmonic number. Prove that

$$\int_0^\infty \frac{\ln \frac{1}{x}}{(1+x)^n}\,\mathrm{d}x = \frac{\mathcal{H}_{n-2}}{n-1}$$

for $\mathbb{N} \ni n > 2$.

3. Let $n, m \in \mathbb{N}$. Prove that

$$\int_0^\infty \frac{\log^n x}{(1+x^2)^m}\,\mathrm{d}x = \mathrm{B}^{(n)}\left(\frac{1}{2}, m-\frac{1}{2}\right) = \frac{1}{m^2\Gamma(m)}\sum_{k=0}^{n}\binom{n}{k}\Gamma^{(k)}\left(\frac{1}{2}\right)\Gamma^{(n-k)}\left(m-\frac{1}{2}\right)$$

where B, Γ are the Euler's Beta and Gamma functions respectively. Here $f^{(n)}$ denotes the n - th derivative of the function f.

4. Let Γ denote the Euler's Gamma function. Prove that:

$$\int_0^1 \ln \Gamma(x+1)\,\mathrm{d}x = \frac{\ln 2\pi}{2} - 1$$

5. Prove that

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n\partial}{n^{2m+1}} = \frac{1}{2}\left[\zeta(2m+1) - \mathrm{Cl}_{2m+1}(2\partial)\right].$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 n\partial}{n^{2m+1}} = \frac{1}{2}\left[\zeta(2m+1) + \mathrm{Cl}_{2m+1}(2\partial)\right].$

Hint: Add them and substract them. Conclude!