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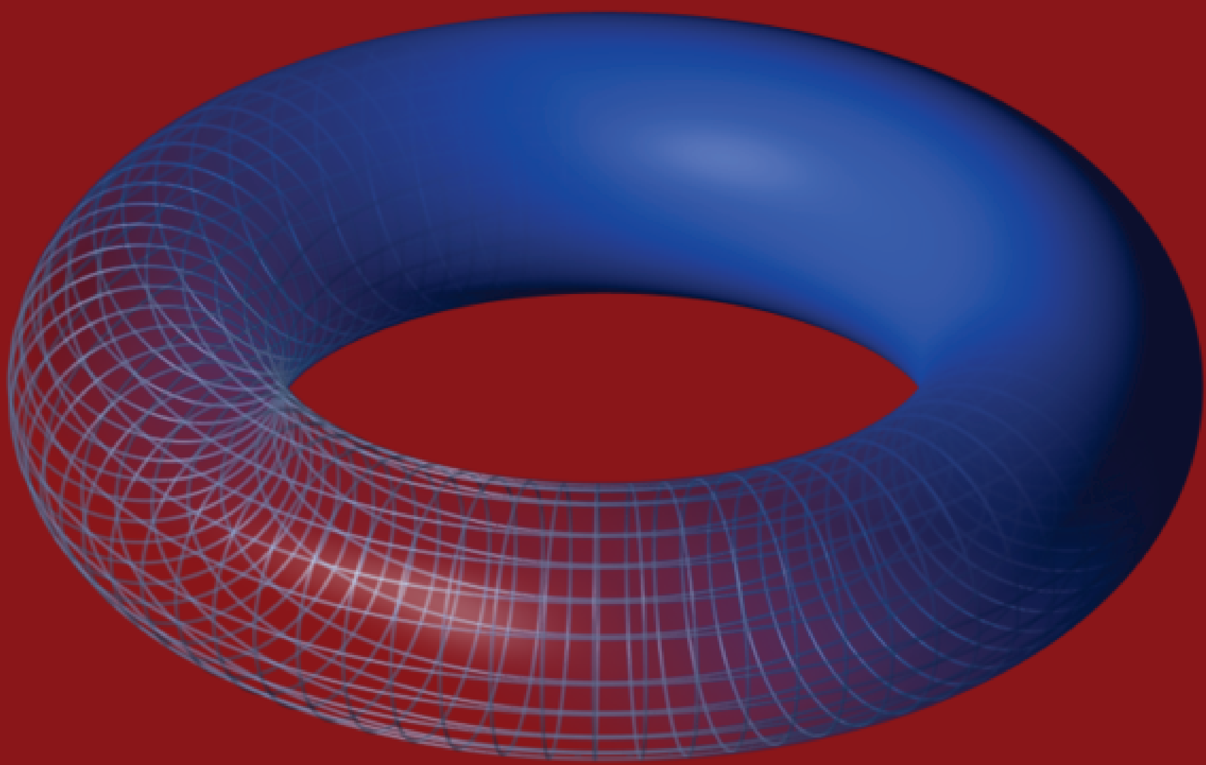

TOLASO NETWORK
2020

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tolaso@tolaso.com.gr

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Editor

Tolaso J. Kos , Farsala, Lárisa, Greece , tolaso@tolaso.com.gr

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The JoM Journal is an electronic mathematical journal which aims at giving the chance to the readers, and the editor himself, to work on interesting mathematical problems or find information about various mathematical topics. The problems presented here are basically a collection of the problems posted on the JoM Blog (hosted at math.tolaso.com.gr) . The level of the topics is undergraduate and beyond. However, there is a section dedicated to inequalities and general mathematics sometimes including mathematical competitions. The JoM journal is consisted of 6 parts:

- Algebra
- Calculus
- Analysis
- Inequalities | General Mathematics
- JoM ... proposes
- JoM ... study

The JoM ... proposes column contains problems that extend the ideas already seen in the previous 4 columns. The JoM study, on the other hand, studies several mathematical concepts. Examples are included whenever necessary. At the end of this part the reader will find problems to exercise himself.

If you want to submit an article at the JoM ... study please contact the author at tolaso@tolaso.com.gr.

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Should you notice any typographical errors , please contact the author so that can be fixed.

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PART 1

Algebra

(1) Let α and β be elements of $\mathcal{S}_n$. Prove that $\alpha^{-1}\beta^{-1}\alpha\beta$ is an even permutation.

Solution. *Let $\sigma : S_n \to \mathbb{Z}_2$ be the sign map, which is ring epimorphism. Then,*

$$\begin{aligned}\sigma(a^{-1}\,\beta^{-1}\,a\,\beta) &= -\sigma(a) - \sigma(b) + \sigma(a) + \sigma(b) \\ &= 0\end{aligned}$$

so the permutation $\alpha^{-1}\beta^{-1}\alpha\beta$ is even.

♦

(2) Let $\mathcal{G}$ be a finite group such that $(|\mathcal{G}|, 3) = 1$. If for the elements $a, \beta \in \mathcal{G}$ holds that:

$$(a\beta)^3 = a^3\beta^3$$

then prove that $\mathcal{G}$ is abelian.

Solution. *Let $x \in \mathcal{G}$ such that $x^3 = e$. If $x \neq e$ then the order of x would be 3. This would immediately imply that the order of x would divide the order of the group $\mathcal{G}$. This is an obscurity due to the data of the exercise. Thus $x = e$. As*

$$(a\beta)^3 = a^3\beta^3$$

we conclude that the mapping $f(x) = x^3$ is an $1-1$ group homomorphism. Therefore forall $a, \beta \in \mathcal{G}$ we have $\beta a\beta a = aa\beta\beta$ or equivelantly $(\beta a)^2 = a^2\beta^2$. Taking advantage of the last relation we get that:

$$\begin{aligned}(a\beta)^4 &= \left((a\beta)^2\right)^2 \\ &= \left(\beta^2 a^2\right)^2 \\ &= \left(a^2\right)^2\left(\beta^2\right)^2 \\ &= a^4\beta^4 \\ &= aaaa\beta\beta\beta\beta\end{aligned}$$

as well as

$$\begin{aligned}(a\beta)^4 &= a\beta a\beta a\beta a\beta \\ &= a\,(\beta a)^3\,\beta \\ &= a\beta^3 a^3\beta \\ &= a\beta\beta\beta aaa\beta\end{aligned}$$

The last two relations hold for all $a, \beta \in \mathcal{G}$. Thus, for all $a, \beta \in \mathcal{G}$ it holds that:

$$aaaa\beta\beta\beta\beta = a\beta\beta\beta aaa\beta$$

which in turn implies

$$f(a\beta) = a^3\beta^3 = \beta^3 a^3 = f(\beta a)$$

and since f is $1-1$ we eventually get $a\beta = \beta a$ proving the claim that $\mathcal{G}$ is abelian.

♦

(3) Let $\mathcal{R}$ be an associative ring with unit and order p^2 where p is a prime number. Prove that $\mathcal{R}$ is commutative.

Solution. *Since $\circ(\mathcal{R}) = p^2$ this means,*

$$\forall r \in \mathcal{R} : \circ(r) \mid \circ(\mathcal{R}) = p^2$$

Therefore, if $r \in \mathcal{R}$ then $\circ(\mathcal{R}) = 1$ or $\circ(\mathcal{R}) = p$ or $\circ(r) = p^2$. If $\exists r \in \mathcal{R}$: $\circ(r) = p^2 = \circ(\mathcal{R})$ then $R = \langle r \rangle$ and therefore $\mathcal{R}$ is commutative. If it is not the case, then consider the cyclic group $\langle 1_R \rangle$. Since $1_\mathcal{R} \neq 0_\mathcal{R}$, $\circ(1_\mathcal{R}) = p$. Let $r \notin \langle 1_R \rangle$. Then $r \neq 0_\mathcal{R}$ and $\circ(r) = p$. Observe that

$$\langle 1_R \rangle \bigcap \langle r \rangle = \{0_\mathcal{R}\}$$

Hence the sum $\langle 1_R \rangle + \langle r \rangle$ is direct and therefore $\circ\left(\langle 1_R \rangle + \langle r \rangle\right) = p^2 = \circ(\mathcal{R})$, which means that $\mathcal{R} = \langle 1_R \rangle + \langle r \rangle$. Now, let $x, y \in \mathcal{R}$. Then

$$x = \kappa 1_\mathcal{R} + \lambda r$$

$$y = \mu 1_\mathcal{R} + \nu r$$

so

$$\begin{aligned} xy &= (\kappa 1_\mathcal{R} + \lambda r)(\mu 1_\mathcal{R} + \nu r) \\ &= \kappa\mu 1_\mathcal{R} + \kappa\nu r + \lambda\mu r + \lambda\nu r^2 \\ &= \mu\kappa 1_\mathcal{R} + \nu\kappa r + \mu\lambda r + \nu\lambda r^2 \\ &= (\mu 1_\mathcal{R} + \nu r)(\kappa 1_\mathcal{R} + \lambda r) \\ &= yx \end{aligned}$$

Hence $\mathcal{R}$ is commutative.

♦

(4) Examine if there is an epimorphism from the dihedral group

$$\mathcal{D}_4 = \langle \sigma, \tau \mid \sigma^4 = \tau^2 = \mathrm{Id}, \tau\sigma\tau = \sigma^3 \rangle$$

onto the additive group $\mathbb{Z}_4$.

Solution. *No there is not. Suppose $\phi : \mathcal{D}_4 \to \mathbb{Z}_4$ is a homomorphism. Then*

$$3\phi(\sigma) = \phi(\sigma^3) = \phi(\tau\sigma\tau) = \phi(\tau) + \phi(\sigma) + \phi(\tau)$$

and so $2\phi(\sigma) = 2\phi(\tau)$. But since $2\phi(\sigma) = 2\phi(\tau) = \phi(\tau^2) = 0$, then $\phi(\sigma), \phi(\tau) \in \{0, 2\}$. So $\phi(\mathcal{D}_4) \subseteq \{0, 2\}$ and so ϕ is not onto.

♦

(5) Which group is isomorphic to the quotient group $(\mathbb{Z}\times\mathbb{Z})/\langle(1,2)\rangle$ where $\langle(1,2)\rangle$ is the normal subgroup of $\mathbb{Z}\times\mathbb{Z}$ generated by $(1,2)$? Justify your answer.

Solution. *The function* $f:\mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}$ *defined as*

$$f((m,n)) = 2m - n$$

is a well defined epimorphism (surjective homomorphism) with

$$\begin{aligned}\ker f &= \{(m,n)\in\mathbb{Z}\times\mathbb{Z} \mid f((m,n)) = 0\}\\ &= \{(m,n)\in\mathbb{Z}\times\mathbb{Z} \mid 2m = n\}\\ &= \{(m,2m)\in\mathbb{Z}\times\mathbb{Z} \mid m\in\mathbb{Z}\}\\ &= \langle(1,2)\rangle\end{aligned}$$

So, by the 1st Isomorphism theorem we have that

$$(\mathbb{Z}\times\mathbb{Z})/\langle(1,2)\rangle \cong \mathbb{Z}$$

(6) Prove that a group $\mathcal{G}$ with $|\mathcal{G}|\le 2^n$, $n\in\mathbb{N}$ can be generated by n elements.

Solution. *Let* $1\ne g_1\in\mathcal{G}$ *and set* $\mathcal{G}_1=\langle g_1\rangle$*. Then* $|\mathcal{G}_1|\ge 2$*. Now, if possible, choose* $g_2\in\mathcal{G}\setminus\mathcal{G}_1$ *and put* $\mathcal{G}_2=\langle g_1,g_2\rangle$*. Then* $\mathcal{G}_2\supseteq\mathcal{G}_1\cup g_2\mathcal{G}_1$ *and therefore*

$$|\mathcal{G}_2|\ge|\mathcal{G}_1\cup g_2\mathcal{G}_1| = |\mathcal{G}_1|+|g_2\mathcal{G}_1| = 2|\mathcal{G}_1|\ge 2^2$$

Again, if possible, choose $g_3\in\mathcal{G}\setminus\mathcal{G}_2$ *and put* $\mathcal{G}_3=\langle g_1,g_2,g_3\rangle$ *Then*

$$|\mathcal{G}_3|\supseteq|\mathcal{G}_2\cup g_3\mathcal{G}_2| = 2|\mathcal{G}_2|\ge 2^3$$

Continuing this way we will eventually find some $r\in\mathbb{N}$ *and* $\mathcal{G}_r=\langle g_1,g_2,\cdots,g_r\rangle$ *with* $|\mathcal{G}_r|\ge 2^r$ *and it is no longer possible to choose any* $g\in\mathcal{G}\setminus\mathcal{G}_r$*. That means* $\langle g_1,g_2,\cdots,g_r\rangle=\mathcal{G}_r=\mathcal{G}$*. But then we'll have* $2^n\ge|\mathcal{G}|=|\mathcal{G}_r|\ge 2^r$ *and so* $r\le n$*. Thus* $\mathcal{G}$ *can be generated by* $r\le n$ *elements.*

♦

(7) Prove that

$$\mathbb{Q}\left(\sqrt{2}+\sqrt{5}+\cdots+\sqrt{n^2+1}\right) = \mathbb{Q}\left(\sqrt{2},\sqrt{5},\cdots,\sqrt{n^2+1}\right)$$

Solution. *Let* $a,b\in\mathbb{Z}_{>0}$ *such that they are not squares of integers. We'll prove that*

$$\mathbb{Q}\left(\sqrt{a},\sqrt{b}\right) = \mathbb{Q}\left(\sqrt{a}+\sqrt{b}\right)$$

It's obvious that $\mathbb{Q}\left(\sqrt{a}+\sqrt{b}\right)\subseteq\mathbb{Q}\left(\sqrt{a},\sqrt{b}\right)$ *since* $\sqrt{a}+\sqrt{b}\in\mathbb{Q}\left(\sqrt{a},\sqrt{b}\right)$*. Suffice to prove that* $\sqrt{a},\sqrt{b}\in\mathbb{Q}\left(\sqrt{a}+\sqrt{b}\right)$*. Note that*

$$\sqrt{a}-\sqrt{b} = \frac{a-b}{\sqrt{a}+\sqrt{b}}\in\mathbb{Q}\left(\sqrt{a}+\sqrt{b}\right)$$

Hence,

$$\sqrt{a} = \frac{\sqrt{a} + \sqrt{b} + \sqrt{a} - \sqrt{b}}{2} \in \mathbb{Q}\left(\sqrt{a} + \sqrt{b}\right) \quad (1)$$

$$\sqrt{b} = \frac{\sqrt{a} + \sqrt{b} - (\sqrt{a} - \sqrt{b})}{2} \in \mathbb{Q}\left(\sqrt{a} + \sqrt{b}\right) \quad (2)$$

Thus, $\mathbb{Q}\left(\sqrt{a}, \sqrt{b}\right) \subseteq \mathbb{Q}\left(\sqrt{a} + \sqrt{b}\right)$. *The result follows. The general case follows from induction on* n.

♦

(8) Let $\mathcal{R}$ be a ring such that:

$$\text{there exists an } n \geq 2 \text{ such that: } a^n = a \text{ forall } a \in \mathcal{R}$$

Prove that $\mathcal{R}$ has no non-zero nilpotent elements. Furthemore, if n is even prove that the characteristic of $\mathcal{R}$ is 2.

Solution. *Let's assume that there is* $x \neq 0$ *such that* $x^m = 0$ *for some integer* $m \geq 2$. *There exists* $k \in \mathbb{N}$ *such that* $n^k > m$, *so he have*

$$x^m = 0 \Rightarrow x^{n^k} = 0 \Rightarrow x^{n^{k-1}} = 0 \Rightarrow \cdots \Rightarrow x^n = 0 \Rightarrow x = 0$$

We got a contradiction so there are no non-zero nilpotent elements in the ring. If n *is even note that* $a = a^n = (-a)^n = -a$ *forall* $a \in \mathcal{R}$ *so* $\text{char}\mathcal{R} = 2$.

♦

(9) Let $(\mathcal{R}, +, \cdot)$ be a ring without necessary a unitary element but it has at least two elements. If $\mathcal{R}$ furthermore satisfies the property: "for every element $a \in \mathcal{R}$ that is not zero $(a \neq 0)$ there exists a unique element $b \in \mathcal{R}$ such that $aba = a$, then prove that:

(i) $\mathcal{R}$ has no non-zero divisors.

(ii) $bab = b$

(iii) $\mathcal{R}$ has a unitary element.

(iv) $\mathcal{R}$ is a divisor ring.

Solution. *(i) Suppose that* $ac = 0$ *and* $a \neq 0$. *We need to show that* $c = 0$. *Since* $a \neq 0$ *there exists a unique* $b \in \mathcal{R}$ *such that* $aba = a$. *But, since* $ac = 0$ *we also have* $a(b + c)a = aba = a$. *Thus, by the uniqueness of* b, *we must have* $b = b + c$ *and so* $c = 0$.

(ii) We have $aba = a$ *and so*

$$a(bab)a = (aba)ba = aba = a$$

Thus, by the uniqueness of b *we must have* $b = bab$.

(iii) *Fix an element* $0 \neq a \in \mathcal{R}$ *and* $b \in \mathcal{R}$ *such that* $aba = a$ *and let* $e = ab$. *Then* $e \neq 0$ *and* $e^2 = e$. *Let* c *be any element of* $\mathcal{R}$. *We claim that* $ce = ec = c$ *which means* $e = 1_{\mathcal{R}}$. *Suppose, to the contrary, that* $ce - c \neq 0$. *Then there exists a unique element* $d \in \mathcal{R}$ *such that* $(ce-c)d(ce-c) = ce-c$. *But, since* $e^2 = e$ *we have* $(ce-c)e(ce-c) = 0$ *and thus* $(ce-c)(d+e)(ce-c) = ce-c$. *Therefore, by the uniqueness of* d *we must have* $d = d+e$ *hence* $e = 0$, *contradiction! So* $ce = c$. *Similarly, we get* $ec = c$.

(iv) *Let* $0 \neq a \in \mathcal{R}$. *We need to show that* a *is invertible. Well, there exists* $b \in \mathcal{R}$ *such that* $aba = a$ *and so, since, by (iii),* $\mathcal{R}$ *has 1, we get* $a(ba-1) = (ab-1)a = 0$. *Hence, since, by (i),* $\mathcal{R}$ *has no (non-zero) zero divisor, we get* $ab = ba = 1$.

10 Let $A, B \in \mathcal{M}_n(\mathbb{C})$. Show that

$$AB - BA = \mathbb{I}_{n\times n}$$

has no solutions.

Solution. *Since* $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ *taking traces on both sides, we have*

$$\mathrm{tr}(AB - BA) = \mathrm{tr}(\mathbb{I}_{n\times n}) \Rightarrow \mathrm{tr}(AB) - \mathrm{tr}(BA) = n \Rightarrow 0 = n$$

♦

PART 2

Calculus

(1) Let J_0 denote the Bessel function of the first kind. Prove that

$$\int_0^\infty J_0(x)\,\mathrm{d}x = 1$$

Solution. *We recall that*

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^n (n!)^2}$$

Hence,

$$\begin{aligned}\int_0^\infty J_0(x) e^{-px}\,\mathrm{d}x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n p^{2n+1}} \binom{2n}{n} \\ &= \frac{1}{p} \sum_{n=0}^{\infty} \binom{-1/2}{n} \frac{1}{p^{2n}} \\ &= \frac{1}{\sqrt{1+p^2}}\end{aligned}$$

Then,

$$\lim_{p\to 0^+} \int_0^\infty J_0(x) e^{-px}\,\mathrm{d}x = \lim_{p\to 0^+} \frac{1}{\sqrt{1+p^2}} = 1$$

Using the fact that the $J_0(x)$ looks like an 'almost periodic' function with decreasing amplitude. If we denote by $\{\alpha_k\}_{k\geq 0}$ the zeros of J_0 then $\alpha_k \nearrow \infty$ as $k \to \infty$ and furthermore

$$\left| \int_{\alpha_k}^\infty J_0(x) e^{-px}\,\mathrm{d}x \right| \leq \int_{\alpha_k}^{\alpha_{k+1}} |J_0(x)| e^{-px}\,\mathrm{d}x \to 0$$

as $k \to \infty$ for each $p \geq 0$. So the integral converges uniformly in this case justifying the interchange of limit and integral. The result follows.

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3343`.

(2) Let $|\alpha| < 1$. Prove that:

$$\prod_{n=0}^{\infty} \left(1 + \alpha^{3n} + \alpha^{2\cdot 3^n}\right) = \frac{1}{1-\alpha}$$

Solution. *The product eventually telescopes;*

$$\begin{aligned}\prod_{n=0}^{\infty} \left(\alpha^{0\cdot 3^n} + \alpha^{1\cdot 3^n} + \alpha^{2\cdot 3^n}\right) &= \prod_{n=0}^{\infty} \frac{1-(\alpha^{3^n})^3}{1-\alpha^{3^n}} \\ &= \prod_{n=0}^{\infty} \frac{1-\alpha^{3^{n+1}}}{1-\alpha^{3^n}}\end{aligned}$$

$$= \frac{1-\alpha^{3^1}}{1-\alpha^{3^0}} \cdot \frac{1-\alpha^{3^2}}{1-\alpha^{3^1}} \cdot \frac{1-\alpha^{3^3}}{1-\alpha^{3^2}} \cdots$$
$$= \frac{1}{1-\alpha}$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3361`*.*

3 Let $s > 2$. Evaluate the series

$$\mathcal{S} = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{m^2+4mn+n^2}{(m^2+mn+n^2)^s}$$

Solution. *Let* $\omega = e^{2\pi i/3}$ *and* $z = m - n\omega$*, then*

$$\sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{m^2+4mn+n^2}{(m^2+mn+n^2)^s} = -\sum_{\substack{z\in\mathbb{Z}[\omega]\\ z\neq 0}} \frac{\overline{\omega}z^2+\omega\overline{z}^2}{(z\overline{z})^s} = 0$$

since the sum over every triple $z, \omega z, \overline{\omega} z$ *vanishes (one should also check that the sum absolutely converges but that's straightforward by abelian summation).*

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3364`*.*

4 ***(Russian Olympiad)*** Evaluate the integral

$$\mathcal{J} = \int_0^1 \frac{\arctan \dfrac{x}{x+1}}{\arctan \dfrac{1+2x-2x^2}{2}} \, \mathrm{d}x$$

Solution. *Let* $f(x) = \dfrac{\arctan \dfrac{x}{x+1}}{\arctan \dfrac{1+2x-2x^2}{2}} = \dfrac{\arctan \dfrac{x}{x+1}}{\arctan \dfrac{1+2x(1-x)}{2}}$. *We note that*

$$\begin{aligned} f(x) + f(1-x) &= \frac{\arctan \dfrac{x}{x+1}}{\arctan \dfrac{1+2x(1-x)}{2}} + \frac{\arctan \dfrac{1-x}{2-x}}{\arctan \dfrac{1+2(1-x)x}{2}} \\ &= \frac{\arctan \dfrac{x}{x+1} + \arctan \dfrac{1-x}{2-x}}{\arctan \dfrac{1+2x-2x^2}{2}} \\ &= 1 \end{aligned}$$

since if we set $u = \arctan \dfrac{x}{x+1}$ *and* $v = \arctan \dfrac{1-x}{2-x}$ *then*

$$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$$

$$= \frac{\dfrac{x}{x+1} + \dfrac{x-1}{x-2}}{1 - \dfrac{x}{x+1} \cdot \dfrac{x-1}{x-2}} = \frac{x^2 - 2x + x^2 - 1}{x^2 - x - 2 - x^2 + x} = \frac{1 + 2x - 2x^2}{2}$$

Hence, $u + v = \arctan \frac{1+2x-2x^2}{2}$. *Therefore,*

$$\begin{aligned}
\mathcal{J} &= \int_0^1 f(x)\,\mathrm{d}x \\
&\overset{y=1-x}{=\!=\!=} \int_0^1 f(1-x)\,\mathrm{d}x \\
&= \frac{1}{2}\left(\int_0^1 f(x)\,\mathrm{d}x + \int_0^1 f(1-x)\,\mathrm{d}x\right) \\
&= \frac{1}{2}\int_0^1 \left(f(x) + f(1-x)\right)\,\mathrm{d}x \\
&= \frac{1}{2}
\end{aligned}$$

♦

5 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let α, β be positive integer numbers such that $\alpha^2 - \beta^2$ is prime. If $\alpha > \beta$ then evaluate the integral

$$\mathcal{J} = \int_{\alpha^2-\beta^2}^{\alpha+\beta} \frac{f^2(x) + f^4(x)}{f^{10}(x) + 1}\,\mathrm{d}x$$

Solution. *Since* $p = \alpha^2 - \beta^2 = (\alpha - \beta)(\alpha + \beta)$ *is prime it follows that either* $\alpha + \beta = 1$ *and* $p = \alpha - \beta$ *or* $\alpha - \beta = 1$ *and* $p = \alpha + \beta$*. The first case is rejected since* $\alpha > \beta$ *hence*

$$\begin{aligned}
\mathcal{J} &= \int_{\alpha^2-\beta^2}^{\alpha+\beta} \frac{f^2(x) + f^4(x)}{f^{10}(x) + 1}\,\mathrm{d}x \\
&= \int_{(\alpha-\beta)(\alpha+\beta)}^{\alpha+\beta} \frac{f^2(x) + f^4(x)}{f^{10}(x) + 1}\,\mathrm{d}x \\
&= \int_{\alpha+\beta}^{\alpha+\beta} \frac{f^2(x) + f^4(x)}{f^{10}(x) + 1}\,\mathrm{d}x \\
&= 0
\end{aligned}$$

♦

6 Evaluate the contour integral

$$\mathcal{J} = \oint\limits_{|z|=1} \cot z\,\mathrm{d}z$$

Solution. *The function has a unique pole at $z = 0$; hence*

$$\begin{aligned}\mathcal{J} &= \oint_{|z|=1} \cot z \, \mathrm{d}z \\ &= 2\pi i \mathfrak{Res}\,(f; z = 0) \\ &= 2\pi i \lim_{z \to 0} \frac{z \cos z}{\sin z} \\ &= 2\pi i\end{aligned}$$

♦

7 Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. Evaluate the integral

$$\mathcal{J} = \int_0^1 (4x^3 - 6x^2 + 8x - 3) g(x^2 - x + 1) \, \mathrm{d}x$$

Solution. *Forget about g! We note that $(x^2 - x + 1)' = 2x - 1$. We further note that*

$$4x^3 - 6x^2 + 8x - 3 = (2x - 1)\left(2x^2 - 2x + 3\right)$$

Hence,

$$\begin{aligned}\mathcal{J} &= \int_0^1 (4x^3 - 6x^2 + 8x - 3) g(x^2 - x + 1) \, \mathrm{d}x \\ &= \int_0^1 \left(2x^2 - 2x + 3\right)(2x - 1)\, g\left(x^2 - x + 1\right) \mathrm{d}x \\ &= \int_0^1 \left(2x^2 - 2x + 2 + 1\right)(2x - 1)\, g\left(x^2 - x + 1\right) \mathrm{d}x \\ &= \int_0^1 \left[2\left(x^2 - x + 1\right) + 1\right](2x - 1)\, g\left(x^2 - x + 1\right) \mathrm{d}x \\ &\overset{u = x^2 - x + 1}{=\!=\!=\!=} \int_1^1 (2u + 1)\, g(u) \, \mathrm{d}u \\ &= 0\end{aligned}$$

♦

8 Evaluate the integral

$$\mathcal{J} = \int_0^2 \frac{\sqrt{x^2 + x + 1} - \sqrt{x^2 - 5x + 7}}{\sqrt{x^2 - x + 1} + \sqrt{x^2 - 3x + 3}} \, \mathrm{d}x$$

Solution. *Applying the substitution $y = 2 - x$ we note that the radicals are switched. Hence, $\mathcal{J} = 0$.* ♦

9 Evaluate the integral

$$\mathcal{J} = \int_0^{\pi/2} \frac{\ln \cos x}{\sin x} \, \mathrm{d}x$$

Solution. *We have successively:*

$$\begin{aligned}
\int_0^{\pi/2} \frac{\ln \cos x}{\sin x}\,\mathrm{d}x &\overset{u=\cos x}{=\!=\!=} \int_0^1 \frac{\ln u}{\sqrt{1-u^2}\sqrt{1-u^2}}\,\mathrm{d}x \\
&= \int_0^1 \frac{\ln u}{1-u^2}\,\mathrm{d}u \\
&= \int_0^1 \ln u \sum_{n=0}^{\infty} u^{2n}\,\mathrm{d}u \\
&= \sum_{n=0}^{\infty} \int_0^1 u^{2n} \ln u \,\mathrm{d}u \\
&= -\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \\
&= -\frac{\pi^2}{8}
\end{aligned}$$

♦

10 Let $\mathcal{H}_n$ denote the n - th harmonic number. Prove that

$$n \int_0^{\infty} x e^{-x} \left(1 - e^{-x}\right)^{n-1} \,\mathrm{d}x = \mathcal{H}_n$$

Solution. *Let I_n denote the LHS. Thus,*

$$\begin{aligned}
I_n &= n \int_0^{\infty} x e^{-x} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} e^{-kx}\,\mathrm{d}x \\
&= n \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \int_0^{\infty} x e^{-(k+1)x}\,\mathrm{d}x \\
&= n \sum_{k=0}^{n-1} \frac{(-1)^k}{(k+1)^2} \binom{n-1}{k} \\
&= \sum_{k=0}^{n-1} \frac{(-1)^k}{k+1} \binom{n}{k+1}
\end{aligned}$$

Thus,

$$\begin{aligned}
I_n - I_{n-1} &= \frac{(-1)^{n-1}}{n} + \sum_{k=0}^{n-2} \frac{(-1)^k}{k+1} \left[\binom{n}{k+1} - \binom{n-1}{k+1} \right] \\
&= \frac{(-1)^{n-1}}{n} + \sum_{k=0}^{n-2} \frac{(-1)^k}{k+1} \binom{n-1}{k} \\
&= \frac{(-1)^{n-1}}{n} + \frac{1}{n} \sum_{k=0}^{n-2} (-1)^k \binom{n}{k+1} \\
&= \frac{(-1)^{n-1}}{n} + \frac{(1-1)^n + 1 - (-1)^{n-1}}{n}
\end{aligned}$$

$$= \frac{1}{n}$$

Since $I_1 = 1$ *we conclude that* $I_n = \mathcal{H}_n$.

♦

3

PART

Analysis

1 Let x_n be a sequence of real numbers , $\mathcal{S} = \sum\limits_{n=1}^{\infty} x_n$ and $\ell = \lim\limits_{n \to +\infty} n x_n$.

(i) Prove that if $\mathcal{S}$ converges and ℓ exists (finite or infinite) then $\ell = 0$.

(ii) Give an example where $x_n > 0$, $\mathcal{S}$ converges but ℓ does not exist.

(iii) Give an example of a decreasing sequence x_n , $\ell = 0$ but $\mathcal{S}$ diverges.

(iv) Prove that if $\mathcal{S}$ converges and x_n is decreasing then

$$\sum_{n=1}^{\infty} n\left(x_n - x_{n+1}\right) = \mathcal{S}$$

Solution. *(i) Let $\mathcal{S}_n$ be the sequence of partial sums , then $\mathcal{S}_n \to \mathcal{S}$. It follows from Cesaro that* $\dfrac{\mathcal{S}_1 + \mathcal{S}_2 + \cdots + \mathcal{S}_n}{n} \to \mathcal{S}$. *Hence*

$$\frac{n x_1 + (n-1)x_2 + \ldots + 2x_{n-1} + x_n}{n} \to \mathcal{S}$$

From the assumption we have that

$$\frac{n+1}{n}\left(x_1 + \cdots + x_n\right) = \frac{n+1}{n}\mathcal{S}_n \to \mathcal{S}$$

Substracting these two we have that

$$\frac{x_1 + 2x_2 + \ldots + (n-1)x_{n-1} + n x_n}{n} \to 0$$

But since $n x_n \to \ell$ it follows from Cesàro and the uniqueness of the limit that the last sum tends to ℓ.

(ii) One such example could be $x_n = \dfrac{1}{n^2}$ *if* $n \neq \frac{1}{2^k}$ *and* $x_{2^n} = \dfrac{1}{2^n}$ *otherwise. Now, the series converges but since $n \cdot \frac{1}{n^2} \to 0$ and $2^n x_{2^n} \to 1$ the sequence $n x_n$ does not converge.*

(iii) The classic example is $x_n = \dfrac{1}{n \ln n}$.

(iv) Since x_n is decreasing it follows that $n x_n \to 0$. Let t_n be the partial sum of the LHS. It follows that

$$\begin{aligned} t_n &= (x_1 - x_2) + 2(x_2 - x_3) + \cdots + n(x_n - x_{n+1}) \\ &= (x_1 + x_2 + \ldots + x_n) - n x_{n+1} \\ &= s_n - \frac{n+1}{n}(n+1)x_{n+1} \\ &\to \mathcal{S} - 0 \end{aligned}$$

The result now follows.

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3367`.

2 Let $\mathcal{H}_n$ denote the n-th harmonic number. Evaluate the limit

$$\ell = \lim_{n \to +\infty}\left(\mathcal{H}_n - \frac{1}{n}\sum_{k=1}^{n}\mathcal{H}_k\right)$$

Solution. *It follows from Cesàro and the definition of the Euler Mascheroni constant that the RHS is equal to*

$$(\mathcal{H}_n - \ln n) - \frac{1}{n}\sum_{k=1}^{n}(\mathcal{H}_k - \ln k) + \ln n - \frac{1}{n}\ln n!$$

The first two terms tend to $\gamma - \gamma = 0$. *All that is left to evaluate the limit of* $\ln n - \frac{\ln n!}{n}$. *For this we have using Riemann sums*

$$\begin{aligned}\lim_{n\to+\infty}\left(\frac{1}{n}\ln n! - \ln n\right) &= \lim_{n\to+\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\ln k - \ln n\right)\\ &= \lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\frac{k}{n}\\ &= \int_0^1 \ln x\,\mathrm{d}x\\ &= -1\end{aligned}$$

Thus $\ell = 1$.

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3376`.

3 Let $\mathbb{N} \ni n \geq 2$. Evaluate the limit

$$\ell = \lim_{x\to 1}\frac{\left(1-\sqrt{x}\right)\left(1-\sqrt[3]{x}\right)\left(1-\sqrt[4]{x}\right)\cdots\left(1-\sqrt[n]{x}\right)}{(1-x)^{n-1}}$$

Solution. *Let us consider the function* $f_n(x) = \sqrt[n]{x}$. *Then,*

$$\frac{1}{n} = f_n'(1) = \lim_{x\to 1}\frac{f_n(x)-f_n(1)}{x-1} = \lim_{x\to 1}\frac{1-\sqrt[n]{x}}{1-x}$$

Hence,

$$\ell = f_2'(1)\cdot f_3'(1)\cdots f_n'(1) = \frac{1}{2}\cdot\frac{1}{3}\cdots\frac{1}{n} = \frac{1}{n!}$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3454`.

4 Evaluate the limit

$$\ell = \lim_{x\to a}\frac{x^a - a^x}{x^x - a^a}$$

Solution. *Rewrite the limit as*

$$\lim_{x\to a}\frac{x^a - a^x}{x^x - a^a} = \lim_{x\to a}\frac{\frac{x^a-a^a}{x-a} - \frac{a^x-a^a}{x-a}}{\frac{x^x-a^a}{x-a}}$$

Using the definition of the derivative we get that limit equals to

$$\ell = \frac{1 - \ln a}{1 + \ln a}$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3457`.

5 Consider the points $A(0,0)$, $B(1,0), C(x,1)$ with $x > 0$. Let $r(x)$ be the radius of the inscribed circle of the triangle ABC.

Prove that

$$\lim_{x \to +\infty} \rho(x) = 0$$

Solution. *Since* $\tan A = \frac{1}{x}$ *we deduce that* $\hat{A} \to 0$ *as* $x \to +\infty$. *Since the incenter* I *lies on the bisector of* A , *it follows that if* D *is the projection of* I *on the* $x'x$ *axis*

$$\rho = AD \tan \frac{A}{2} \leq 1 \tan \frac{A}{2} \to 0$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3460`.

6 Given a function $f : [-1,1] \to [0,\pi]$ such that

$$\cos f(x) = x \quad \text{for all} \quad x \in [-1,1] \tag{1}$$

(i) Evaluate $f(0)$.

(ii) Prove that f is one to one.

(iii) Prove that $f(\cos x) = x$ for all $x \in [0,\pi]$.

(iv) Find the range of f.

(v) Sketch the graph of f.

Solution. *(i) Setting* $x = 0$ *at* (1) *we have that*

$$\cos f(0) = 0 \Leftrightarrow x = \frac{\pi}{2} + \kappa\pi \;,\; \kappa \in \mathbb{Z}$$

However,

$$\begin{aligned} 0 \leq f(0) \leq \pi &\Rightarrow 0 \leq \kappa\pi + \frac{\pi}{2} \leq \pi \\ &\Rightarrow -\frac{\pi}{2} \leq \kappa\pi \leq \frac{\pi}{2} \\ &\Rightarrow -\frac{1}{2} \leq \kappa \leq \frac{1}{2} \\ &\xRightarrow{\kappa \in \mathbb{Z}} \kappa = 0 \end{aligned}$$

Thus $f(0) = \frac{\pi}{2}$.

(ii) *Let* $x_1, x_2 \in [-1, 1]$ *such that* $f(x_1) = f(x_2)$. *Thus,*

$$\begin{aligned} f(x_1) = f(x_2) &\Leftrightarrow \cos f(x_1) = \cos f(x_2) \\ &\Leftrightarrow x_1 = x_2 \end{aligned}$$

Hence f *is* $1-1$.

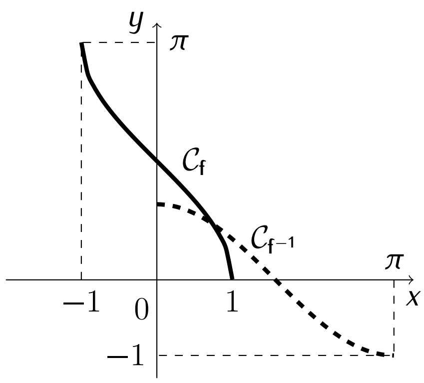
(iii) *Setting* $x = \cos x$ *we have*

$$\begin{aligned} \cos f(x) = x &\Leftrightarrow \cos f(\cos x) = \cos x \\ &\Leftrightarrow f(\cos x) = x \end{aligned}$$

since $\cos$ *is strictly decreasing in* $[0, \pi]$.

(iv) $f([-1, 1]) = [0, \pi]$ *because* $f^{-1}(x) = \cos x$, $x \in [0, \pi]$.

(v) *The graph is seen at the following figure:*



♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3478`.

7 Let

$$f(x) = \frac{\sqrt{1+2x} \cdot \sqrt[4]{1+4x} \cdot \sqrt[6]{1+6x} \cdots \sqrt[100]{1+100x}}{\sqrt[3]{1+3x} \cdot \sqrt[5]{1+5x} \cdot \sqrt[7]{1+7x} \cdots \sqrt[101]{1+101x}}$$

Evaluate $f'(0)$.

Solution. *The domain of* f *is* $\mathcal{A}_f = \left(-\frac{1}{101}, +\infty\right)$. *Let us consider the logarithmic of* f

$$g(x) = \ln f(x) = \sum_{k=2}^{101} \frac{(-1)^k}{k} \ln(1+kx)$$

and differentiate it; Hence,

$$g'(x) = \frac{f'(x)}{f(x)} = \sum_{k=2}^{101} \frac{(-1)^k}{1+kx}$$

For $x = 0$ *we have*

$$g'(0) = \sum_{k=2}^{101} (-1)^k = 0$$

Thus, $f'(0) = 0$ *since* $f(0) = 1$.

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3494`.

8 Let $f, g : \mathbb{R} \to \mathbb{R}$ be two continuous functions such that

$$\int_{\int_x^1 f(t)\,dt}^{\int_0^x f(t)\,dt} g(t)\,dt > 0 \quad \text{for all } \; x \in \mathbb{R} \setminus \{0, 1\}$$

If $g(x) + g(2 - x) = 2$ and $g(x) \neq 0$ for all $x \in \mathbb{R}$, then prove that:

(i) $\displaystyle\int_0^x f(t)\,dt > \int_x^1 f(t)\,dt$

(ii) $\displaystyle\int_0^1 f(t)\,dt = 0$

(iii) $f(1) = f(0) = 0$

(iv) the equation $\displaystyle f(x) \int_x^1 f(t)\,dt = f(x)f'(x)$ has at least a root in $(0, 1)$ if f is differentiable.

(v) the area of g bounded by the axis $x'x$ and the lines $x = 0$, $x = 2$ is 2 square meters.

Solution. *(i) Since* $g(x) \neq 0$ *for all* $x \in \mathbb{R}$ *it follows that the sign of* g *is constant. For* $x = 1$ *we get that* $g(1) = 1 > 0$ *hence* $g(x) > 0$ *for all* $x \in \mathbb{R}$. *Therefore,*

$$\int_{\int_x^1 f(t)\,dt}^{\int_0^x f(t)\,dt} g(t)\,dt > 0 \overset{g(x)>0}{\Longleftrightarrow} \int_0^x f(t)\,dt > \int_x^1 f(t)\,dt$$

(ii) Set $\mathrm{F}(x) = \int_0^x f(t)\,dt$ *and* $\mathrm{G}(x) = \int_x^1 f(t)\,dt$. *Since* $\mathrm{F}(x) > \mathrm{G}(x)$ *taking limits we have*

$$\lim_{x\to 0} \mathrm{F}(x) \geq \lim_{x\to 0} \mathrm{G}(x) \Leftrightarrow \int_0^1 f(t)\,dt \leq 0 \tag{1}$$

$$\lim_{x\to 1} \mathrm{F}(x) \geq \lim_{x\to 1} \mathrm{G}(x) \Leftrightarrow \int_0^1 f(t)\,dt \geq 0 \tag{2}$$

Combining (1) , (2) *we get the that* $\displaystyle\int_0^1 f(t)\,dt = 0$.

(iii) Consider the differentiable function

$$h(x) = \int_0^x f(t)\,dt - \int_x^1 f(t)\,dt = \int_0^x f(t)\,dt + \int_1^x f(t)\,dt \geq 0$$

We note that $h(1) = h(0) = 0$ and $h(x) \geq 0 = h(1) = h(0)$ hence by Fermat's theorem it follows that $h'(0) = h'(1) = 0$. On the other hand,

$$h'(x) = 2f(x)$$

The result follows.

(iv) *It follows by Rôlle's theorem at the function* $\mathrm{H}(x) = \mathrm{G}^2(x) + f^2(x)$ *at* $[0, 1]$.

(v) *It follows by Rôlle's theorem at the function* $\mathrm{K}(x) = x\mathrm{G}(x)$ *at* $[0, 1]$.

(vi) *We have successively*

$$\begin{aligned}\mathcal{A} &= \int_0^2 |g(x)| \, \mathrm{d}x \\ &\overset{g(x)>0}{=\!=\!=} \int_0^2 g(x) \, \mathrm{d}x \\ &\overset{y=2-x}{=\!=\!=} \int_0^2 g(2-x) \, \mathrm{d}x \\ &= \frac{1}{2} \int_0^2 (g(x) + g(2-x)) \, \mathrm{d}x \\ &= \frac{1}{2} \int_0^2 2 \, \mathrm{d}x \\ &= 2\end{aligned}$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3484`.

(9) ***(Huygen's Inequality)*** Let $\alpha \in \left[0, \frac{\pi}{2}\right)$. Prove that

$$2\sin\alpha + \tan\alpha \geq 3\alpha$$

Solution. *Let $f(\alpha) = 2\sin\alpha + \tan\alpha$. f is twice differentiable with*

$$f''(\alpha) = -2\sin\alpha + 2\tan\alpha + 2\tan^3\alpha \geq 0$$

Hence f is convex. The tangent at $(0, 0)$ has equation $y = 3x$. The result follows.

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3503`.

(10) For $v = \langle x_1, x_2, \ldots, x_n \rangle \in \mathbb{R}^n$ we define $\|v\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{1/p}$ and $\|v\|_\infty = \max_{1 \leq i \leq n} |x_i|$. For which v does the series

$$\sum_{p=1}^{\infty} \left(\|v\|_p - \|v\|_\infty \right)$$

converge?

Solution. *We may assume that v is not the zero vector and $n > 1$ otherwise the series is trivially convergent. Then, we show that the series is convergent if and only if there is exactly one component of maximal absolute value.*

- *If the above condition is satisfied then, without loss of generality, let x_1 be the component of maximal absolute value and let $t = \dfrac{\max\limits_{2\le i\le n} |x_i|}{|x_i|} \in [0,1]$. Hence , as $p \to +\infty$*

$$\begin{aligned} 0 &\le \|v\|_p - \|v\|_\infty \\ &= \|v\|_\infty \left((1 + (n-1)\, t^p)^{1/p} - 1 \right) \\ &= \|v\|_\infty \left(\exp\left(\frac{\ln(1 + (n-1)t^p)}{p} \right) - 1 \right) \\ &\sim \|v\|_\infty (n-1) \frac{t^p}{p} \end{aligned}$$

and the given series is convergent because $\displaystyle\sum_{p=1}^{\infty} \frac{t^p}{p} < +\infty$.

- *If the above condition is not satisfied, then there are at least 2 components of maximal absolute value and therefore*

$$\begin{aligned} \|v\|_p - \|v\|_\infty &\ge \|v\|_\infty \left(2^{1/p} - 1\right) \\ &= \|v\|_\infty \left(\exp\left(\frac{\ln 2}{p} \right) - 1 \right) \\ &\sim \|v\|_\infty \frac{\ln 2}{p} \end{aligned}$$

and the given series is not convergent because $\displaystyle\sum_{p=1}^{\infty} \frac{1}{p} = +\infty$.

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3509`*.*

11 (i) Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{|\cos n|}{n}$ diverges.

(ii) Examine the convergence of the series

$$\mathcal{S} = \sum_{n=1}^{\infty} \left(\sqrt[n]{n} - 1 \right) \cos n$$

Does the series converge absolutely?

Solution. *(i) First of all observe that*

$$\begin{aligned} |\cos n| + |\cos(n+1)| &\ge |\sin(n+1)\cos n| + |\cos(n+1)\sin n| \\ &\ge |\sin(n+1)\cos n - \cos(n+1)\sin n| \end{aligned}$$

$$= |\sin(n+1-n)|$$
$$= |\sin 1|$$

Hence,

$$\begin{aligned}
\sum_{n=1}^{2N} \frac{|\cos n|}{n} &= \sum_{n=1}^{N} \left(\frac{|\cos(2n-1)|}{2n-1} + \frac{|\cos(2n)|}{2n} \right) \\
&\geq \sum_{n=1}^{N} \frac{|\cos(2n-1)| + |\cos(2n)|}{2n} \\
&\geq \frac{|\sin 1|}{2} \sum_{n=1}^{N} \frac{1}{n}
\end{aligned}$$

thus the series diverges.

(ii) The series $\mathcal{S}$ does not converge absolutely since

$$\sqrt[n]{n} - 1 = e^{\frac{\ln n}{n}} - 1 \geq \left(1 + \frac{1}{n} \ln n\right) - 1 = \frac{1}{n} \ln n \geq \frac{1}{n}$$

Thus,

$$\sum_{n=1}^{\infty} \left| \left(\sqrt[n]{n} - 1\right) \cos n \right| = \sum_{n=1}^{\infty} \left(\sqrt[n]{n} - 1\right) |\cos n| \geq \sum_{n=1}^{\infty} \frac{|\cos n|}{n}$$

The series on the other hand converges. That is because the sequence $\sqrt[n]{n}$ is decreasing (from $n \geq 3$ and beyond) and approaches 1. Hence $\sqrt[n]{n} - 1 \to 0$. Furthermore , the partial sums of $\sum \cos n$ are known to be bounded. Hence, Dirichlet's criterion yields the result.

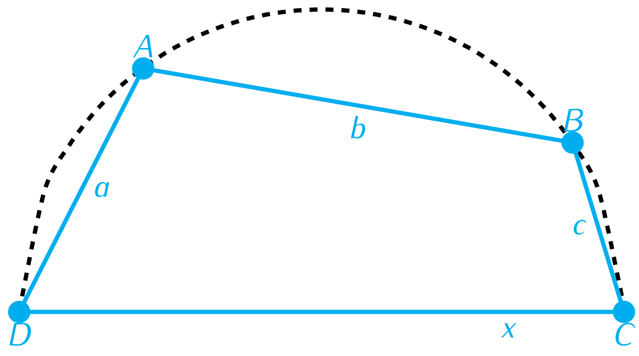
♦

4

PART

General Mathematics

1 Given a cyclic quadrilateral $ABCD$ inscribed in a semicircle of diameter CD as shown at the figure with $CD = x$ and sides of lengths a, b, c and x



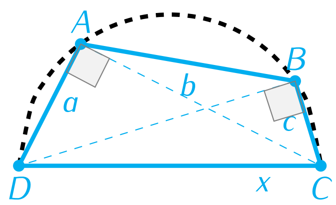
show that:

$$x^3 - \left(a^2 + b^2 + c^2\right) x - 2abc = 0$$

Solution. *We state the following theorem:*

Theorem (Ptolemy's theorem)

If a quadrilateral is inscribable in a circle then the product of the lengths of its diagonals is equal to the sum of the products of the lengths of the pairs of opposite sides.

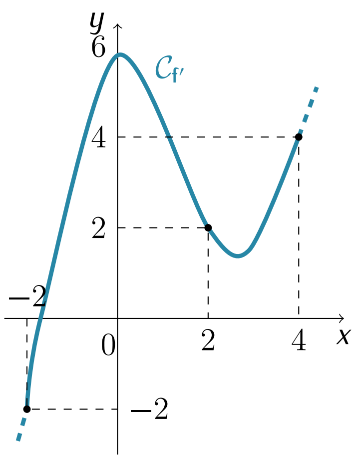


Applying Pythagoras' Theorem to both ADC and DBC along with Ptolemy's Theorem we get the result.

♦

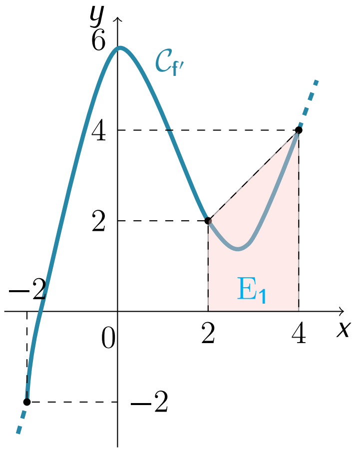
Exercise lies in `https://www.math.tolaso.com.gr/?p=3381`.

2 The following figure depicts the graph of the derivative of f.



Consider the function $g(x) = 2f(x) - x^2$. Order the numbers $g(-2), g(2), g(4)$.

Solution. *We are working on the following figure.*



The area included by f' *, the axis* $x'x$ *and the lines* $x = 2$ *,* $x = 4$ *is less than* E_1*. Hence,*

$$\begin{aligned}\int_2^4 |f'(x)|\,\mathrm{d}x < \mathrm{E}_1 &\Leftrightarrow \int_2^4 f'(x)\,\mathrm{d}x < \frac{(4+2)\cdot 2}{2}\\ &\Leftrightarrow f(4) - f(2) < 6\\ &\Leftrightarrow f(2) < f(4) - 6\end{aligned}$$

On the other hand $\int\limits_{-2}^{4} f'(x)\,\mathrm{d}x > 6$*. Hence,*

$$\begin{aligned}\int_{-2}^4 f'(x)\,\mathrm{d}x > 6 &\Leftrightarrow f(4) - f(-2) > 6\\ &\Leftrightarrow f(-2) < f(4) - 6\end{aligned}$$

Hence,

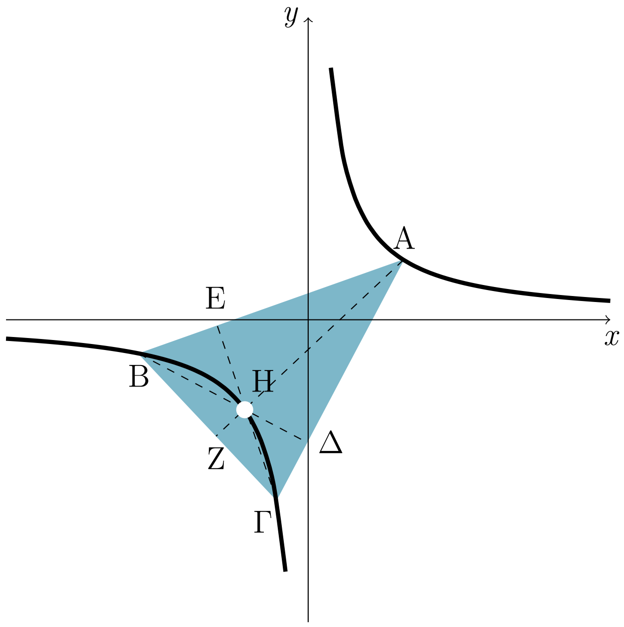
$$\begin{aligned}f(-2) < f(4) - 6 < f(2) &\Leftrightarrow 2f(-2) < 2f(4) - 12 < 2f(2)\\ &\Leftrightarrow 2f(-2) - 4 < 2f(4) - 16 < 2f(2) - 4\\ &\Leftrightarrow g(-2) < g(4) < g(2)\end{aligned}$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3406`*.*

(3) The vertices of a triangle lie on the hyperbola $y = \frac{1}{x}$. Prove that its orthocentre also lies on the hyperbola.

Solution. *We are working on the following figure*



Let $\mathrm{A}\left(p, \frac{1}{p}\right)$, $\mathrm{B}\left(q, \frac{1}{q}\right)$ *and* $\Gamma\left(r, \frac{1}{r}\right)$. *Let us denote as* H *its orthocentre. We have that:*

$$\lambda_{\mathrm{B}\Gamma} = \frac{\frac{1}{r} - \frac{1}{q}}{r - q} = -\frac{1}{qr}$$

Hence, the slope of the altitude AZ *is* qr. *Similarly,*

$$\lambda_{\mathrm{AB}} = \frac{\frac{1}{q} - \frac{1}{p}}{q - p} = -\frac{1}{pq}$$

Hence, the slope of the altitude ΓE *is* pq. *Hence,*

$$(\varepsilon)_{\mathrm{B}\Gamma} : y - \frac{1}{p} = qr\left(x - p\right)$$

and

$$(\varepsilon)_{\Gamma\mathrm{E}} : y - \frac{1}{r} = pq\left(x - r\right)$$

Solving this linear system we have

$$\begin{aligned}(\varepsilon)_{\mathrm{B}\Gamma} = (\varepsilon)_{\Gamma\mathrm{E}} &\Leftrightarrow qr\left(x - p\right) + \frac{1}{p} = \frac{1}{r} + pq\left(x - r\right) \\ &\Leftrightarrow x = -\frac{1}{pqr}\end{aligned}$$

and finally $y = -pqr$. *So,* $\mathrm{H}\left(-\frac{1}{pqr}, -pqr\right)$. *This proves the claim.*

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3471`.

(4) Let σ denote the divisor function. Prove that

$$\lim_{n\to+\infty}\frac{1}{n^2}\sum_{k=1}^{n}\sigma(k)=\frac{\pi^2}{12}$$

Solution. *We have successively:*

$$\begin{aligned}\frac{1}{n^2}\sum_{k\leq n}\sigma(k) &= \frac{1}{n^2}\sum_{k\leq n}\sum_{q\mid k} q\\ &= \frac{1}{n^2}\sum_{q,d\mid qd\leq n} q\\ &= \frac{1}{n^2}\sum_{d\leq n}\sum_{q\leq n/d} q\\ &= \frac{1}{2n^2}\sum_{d\leq n}\left(\frac{n^2}{d^2}+\mathcal{O}\left(\frac{n}{d}\right)\right)\\ &= \frac{1}{2}\sum_{d\leq n}\frac{1}{d^2}+\mathcal{O}\left(\frac{1}{n}\sum_{d\mid n}\frac{1}{d}\right)\\ &= \frac{1}{2}\sum_{d\leq n}\frac{1}{d^2}+\mathcal{O}\left(\frac{\ln n}{n}\right)\\ &\to \frac{\pi^2}{12}\end{aligned}$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3499`*.*

(5) Let $\mathbb{R}^n$ be endowed with the usual product and the usual norm. If $v=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ then we define $\sum v = x_1+x_2+\cdots+x_n$. Prove that

$$\|v\|^2\,\|w\|^2\geq (v\cdot w)^2+\frac{\left(\|v\|\,\left|\sum w\right|-\|w\|\,\left|\sum v\right|\right)^2}{n}$$

Solution. *We will show the more general inequality*

$$\left(\|v\|^2\,\|w\|^2-(v\cdot w)^2\right)\|u\|^2\geq\left\|(w,u)\,v-(v,u)\,w\right\|^2$$

where $u\in\mathbb{R}^n$. Taking $u=(1,1,\ldots,1)$ we get the requested inequality. If v,w are linearly dependent then $\|v\|^2\,\|w\|^2=(v\cdot w)^2$ the inequality holds. We assume now that v and w are linearly independent. Then $u=\alpha v+\beta w+z$ where $\alpha,\beta\in\mathbb{R}$ and $z\perp v$, $z\perp w$. Moreover,

$$\begin{cases}(v,u) &= \alpha\,\|v\|^2+\beta\,(v,w)\\ (w,u) &= \alpha\,(v,w)+\beta\,\|w\|^2\end{cases}$$

and by solving the linear system we find

$$\alpha=\frac{(v,u)\,\|w\|^2-(w,u)\,(v,w)}{\|v\|^2\,\|w\|^2-(v,w)^2}\quad,\quad\beta=\frac{(w,u)\,\|u\|^2-(v,u)\,(v,w)}{\|v\|^2\,\|w\|^2-(v,w)^2}$$

Hence,

$$\begin{aligned}
\left(\|v\|^2\|w\|^2-(v\cdot w)^2\right)\|u\|^2 &= \left(\|v\|^2\|w\|^2-(v\cdot w)^2\right)\left(\left\|\alpha v+\beta w\right\|^2+\|z\|^2\right)\\
&\geq \left(\|v\|^2\|w\|^2-(v\cdot w)^2\right)\left(\left\|\alpha v+\beta w\right\|^2\right)\\
&= \left(\|v\|^2\|w\|^2-(v\cdot w)^2\right)\left(\alpha^2\|v\|^2+\beta^2\|w\|^2+2\alpha\beta\,(v,w)\right)\\
&= (w,u)^2\|v\|^2+(v,u)\|w\|^2-2\,(w,u)\,(v,u)\,(v,w)\\
&= \left\|(w,u)\,v-(v,u)\,w\right\|^2
\end{aligned}$$

♦

Exercise lies in `https://www.math.tolaso.com.gr/?p=3507`.

(6) Let lcm denote the least common divisor. Prove that the number:

$$\mathcal{I}=\sum_{n=1}^{\infty}\frac{1}{\operatorname{lcm}(1,2,\ldots,n)}$$

is irrational.

Solution. *We argue by contradiction. Suppose contrary that $S=\frac{a}{b}$ for some relatively prime positive integers a and b. Let $p_1,p_2,p_3,\ldots$ be the increasing sequence of all prime natural numbers greater than b. Using Bertrand's Postulate,*

$$p_r<p_{r+1}<2\,p_r$$

for all $r=1,2,3,\ldots$. Thus,

$$p_{r+1}-p_r\leq p_r-1$$

for each positive integer r. Note that, for infinitely many positive integers r, it holds that $p_r\equiv 2 \pmod 3$. Therefore, the equality $p_{r+1}-p_r=p_r-1$ does not happen (or else, $p_{r+1}\equiv 0 \pmod 3$). Hence, $p_{r+1}-p_r<p_r-1$ for infinitely many such r.

Now,

$$\begin{aligned}
L_{p_1-1}\left(S-\sum_{k=1}^{p_1-1}\frac{1}{L_k}\right) &\leq \sum_{r=1}^{\infty}\sum_{k=p_r}^{p_{r+1}-1}\frac{L_{p_1-1}}{L_k}\leq\sum_{r=1}^{\infty}\sum_{k=p_r}^{p_{r+1}-1}\frac{1}{p_1p_2\cdots p_r}\\
&=\sum_{r=1}^{\infty}\frac{p_{r+1}-p_r}{p_1p_2\cdots p_r}<\sum_{r=1}^{\infty}\frac{p_r-1}{p_1p_2\ldots p_r}\\
&=\left(1-\frac{1}{p_1}\right)+\left(\frac{1}{p_1}-\frac{1}{p_1p_2}\right)+\left(\frac{1}{p_1p_2}-\frac{1}{p_1p_2p_3}\right)+\ldots\\
&=1\,.
\end{aligned}$$

This is a contradiction, as $b\mid L_{p-1}$ and $S>\sum\limits_{k=1}^{p_1-1}\frac{1}{L_k}$, which means $L_{p-1}\left(S-\sum\limits_{k=1}^{p_1-1}\frac{1}{L_k}\right)$ is a positive integer. Therefore, S cannot be a rational number.

♦

7 Prove that the number $\underbrace{111\cdots111}_{91}$ is composite.

Solution. *Note that*

$$\underbrace{111\cdots111}_{91} = \underbrace{\underbrace{1111111}_{7}\underbrace{1111111}_{7}\cdots\underbrace{1111111}_{7}}_{13}$$
$$= \left(1\underbrace{\underbrace{0000001}\underbrace{0000001}\cdots\underbrace{0000001}}_{13}\right)\cdot\underbrace{1111111}_{7}$$

♦

PART 5

JoM ... proposes

1. Let p be a prime and let $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Find all $p \times p$ matrices A and B over $\mathbb{F}_p$ such that $AB - BA = \mathbb{I}$.

   Question: Can you do that in characheristic zero or for $n \times n$ matrices where p does not divide n ? Give a brief explanation.

2. Let A be a complex matrix with real determinant. Prove that A is the product of 4 Hermitian matrices.

3. Let $A \in \mathcal{M}_n(\mathbb{R})$ be an invertible matrix. Prove that

$$\det A = \frac{1}{n!}\begin{vmatrix} \operatorname{tr}(A) & 1 & 0 & 0 & \cdots & 0 \\ \operatorname{tr}(A^2) & \operatorname{tr}(A) & 2 & 0 & \cdots & 0 \\ \operatorname{tr}(A^3) & \operatorname{tr}(A^2) & \operatorname{tr}(A) & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \operatorname{tr}(A^{n-1}) & \operatorname{tr}(A^{n-2}) & \cdots & \cdots & \operatorname{tr}(A) & n-1 \\ \operatorname{tr}(A^n) & \operatorname{tr}(A^{n-1}) & \operatorname{tr}(A^{n-2}) & \cdots & \cdots & \operatorname{tr}(A) \end{vmatrix}$$

4. Let $n \in \mathbb{N}$ and $1 \le k \le n$. Evaluate the determinant

$$\mathcal{D} = \begin{vmatrix} \binom{n}{0} & \binom{n}{1} & \cdots & \binom{n}{k} \\ \binom{n+1}{0} & \binom{n+1}{1} & \cdots & \binom{n+1}{k} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{n+k}{0} & \binom{n+k}{1} & \cdots & \binom{n+k}{k} \end{vmatrix}$$

5. Prove that $\displaystyle\int_0^\infty \frac{\arctan x - xe^{-x}}{x^2}\,\mathrm{d}x = 1 + \gamma$ where γ denotes the Euler – Mascheroni constant.

6. Let $a, b > 0$ and $d = \gcd(a, b)$. Prove that

$$\int_0^1 \left(\{ax\} - \frac{1}{2}\right)\left(\{bx\} - \frac{1}{2}\right)\mathrm{d}x = \frac{d^2}{12ab}$$

7. Let $n \in \mathbb{N}$. Prove that

$$\int_0^1 \frac{\ln^n \frac{1+x}{1-x}}{\sqrt{1-x^2}}\frac{\mathrm{d}x}{1+x} = 2^n\Gamma(n+1)$$

   where Γ denotes the Euler's Gamma function.

8. Let $\alpha \in \mathbb{R}$. Prove that $\displaystyle\sum_{n=1}^{\infty} 2^{2n}\sin^4\frac{a}{2^n} = a^2 - \sin^2 a$.

9. Let r_n be a sequence of all rational numbers in $(0, 1)$. Show that the series $\displaystyle\sum_{n=2}^{\infty} |r_n - r_{n-1}|$ diverges.

10. Let $\gamma_n = \mathcal{H}_n - \ln n$. Evaluate the limit $\ell = \lim\limits_{n\to+\infty} n(\gamma_n - \gamma)$.

11. For $\alpha \in \left(0, \frac{\pi}{2}\right)$ prove that $\alpha > \dfrac{3\sin\alpha}{2+\cos\alpha}$.

12 For $x \geq 0$ prove that $2\sinh x + \tanh x \geq 3x$.

13 Prove that in any triangle the following inequality $\sum \tan^2 \frac{A}{2} \geq 2 - 8 \prod \sin \frac{A}{2}$ holds.

14 Prove the following double inequality, where the sum and product are cyclic over the angles A, B, C of a triangle

$$\sum \sin^2 A \leq 2 + 16 \prod \sin^2 \frac{A}{2} \leq \frac{9}{4}$$

15 Prove that in any triangle ABC the following double inequality

$$\frac{4}{9} \sum \sin B \sin C \leq \prod \cos \frac{B-C}{2} \leq \frac{2}{3} \sum \cos A$$

holds.

!

Solutions should be submitted at the e-mail tolaso@tolaso.com.gr and will be published in the next edition. Solutions should be submitted by February 28 , 2021.

Solutions should be written in LaTeX. Code should be clear and easy reading. Finally, all solutions should be concise.

6

PART

JoM ... study

Author: Tolaso

JoM ... studies functions and series

Lemma

Let $f:\mathbb{N}^*\to\mathbb{N}^*$ be $1-1$. The series

$$\sum_{n=1}^{\infty}\frac{f(n)}{n^2}$$

diverges.

Proof. **1st proof:** Observe that of $\{f(N+1), f(N+2),\cdots, f(3N)\}$ only N of them are possibly at most N. The rest of them must be strictly greater than N. This yields the estimate

$$\sum_{n=N+1}^{3N}\frac{f(n)}{n^2}\geq\frac{1}{(3N)^2}\sum_{n=N+1}^{3N}f(n)>\frac{1}{9N^2}\cdot N\cdot N=\frac{1}{9}$$

It follows from Cauchy's criterion that the series diverges.

2nd proof: Since f is a permutation of $\mathbb{N}$ it follows that

$$f(1)+f(2)+\cdots+f(n)\geq 1+2+\cdots+n=\frac{n(n+1)}{2}$$

Thus, by Abel's summation formula, we have

$$\begin{aligned}\sum_{n=1}^{N}\frac{f(n)}{n^2}&=\sum_{n=1}^{N-1}(f(1)+\cdots+f(n))\left(\frac{1}{n^2}-\frac{1}{(n+1)^2}\right)+\frac{1}{N^2}\sum_{n=1}^{N}f(n)\\&\geq\sum_{n=1}^{N-1}\frac{n(n-1)}{2}\cdot\frac{2n+1}{n^2(n+1)^2}+\frac{N+1}{2N}\\&=\sum_{n=1}^{N-1}\frac{(n-1)(2n+1)}{n(n+1)^2}+\frac{N+1}{2N}\\&\geq\sum_{n=1}^{N-1}\frac{(n-1)}{n(n+1)}+\frac{1}{2}\end{aligned}$$

The result follows.

♦

Author: Tolaso

JoM ... studies a Fejér integral

Theorem

Let $n \in \mathbb{N}$. It holds that

$$\int_0^{\pi} \left(\frac{\sin nx}{\sin x} \right)^2 \, \mathrm{d}x = n\pi$$

Proof. ***1st proof:*** We note that

$$\begin{aligned} \frac{\sin nx}{\sin x} &= \frac{e^{inx} - e^{-inx}}{e^{ix} - e^{-ix}} \\ &= e^{-i(n-1)x} \frac{\left(e^{2ix}\right)^n - 1}{e^{2ix} - 1} \\ &= e^{-i(n-1)x} \sum_{k=0}^{n-1} e^{2ikx} \end{aligned}$$

This shows that the Fourier series

$$\sum_{k=-\infty}^{\infty} a_k e^{ikx} = \frac{\sin nx}{\sin x} = e^{-i(n-1)x} \sum_{k=0}^{n-1} e^{2ikx}$$

has exactly n coefficients that are 1 and all others vanish. It follows from Parseval's that

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{\sin^2 nx}{\sin^2 x} \, \mathrm{d}x = \sum_{k=-\infty}^{\infty} |a_k|^2 = n$$

Due to symmetry the initial integral evaluates at half. This proves the result.

2nd proof: Making use of the identity $2\sin^2 x = 1 - \cos 2x$ we get that

$$\begin{aligned} \int_0^{\pi} \frac{\sin^2 nx}{\sin^2 x} \, \mathrm{d}x &= \int_0^{\pi} \frac{1 - \cos 2nx}{1 - \cos 2x} \, \mathrm{d}x \\ &\overset{u=2x}{=\!=\!=} \frac{1}{2} \int_0^{2\pi} \frac{1 - \cos nx}{1 - \cos x} \, \mathrm{d}x \\ &= \frac{1}{2} \int_0^{2\pi} \frac{2 - \left(e^{inx} + e^{-inx}\right)}{2 - \left(e^{ix} + e^{-ix}\right)} \, \mathrm{d}x \\ &= \frac{1}{2} \oint_{\gamma} \frac{2 - \left(z^n + z^{-n}\right)}{2 - \left(z + z^{-1}\right)} \frac{\mathrm{d}z}{iz} \end{aligned}$$

where $\gamma : [0, 2\pi] \to \mathbb{R}$ the curve defined as $\gamma(\theta) = e^{i\theta}$. On the other hand,

$$\frac{2 - \left(z^n + z^{-n}\right)}{2 - \left(z + z^{-1}\right)} \frac{1}{z} = \frac{(z^n - 1)^2}{z^n (z-1)^2} = \frac{(1 + z + \cdots + z^{n-1})^2}{z^n}$$

Furthermore , the coefficient of $z^{\nu-1}$ at the expansion $(1 + z + \cdots + z^{n-1})^2$ equals n. It follows from Cauchy's Residue theorem that the integral equals $n\pi$.

♦

Author: Tolaso

JoM ... studies analytic series

Theorem 1

Let μ denote the Möbius function. It holds that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0$$

Proof. The proof is comparable in difficulty to the Prime Number Theorem and technical. So , it is omitted! ♦

Theorem 2

Let $\sigma(n)$ denote the sum of of all divisors of n; $\sigma(\mathrm{n}) = \sum_{\mathrm{d}|\mathrm{n}} \mathrm{d}$. For all $s \in \mathbb{R} \mid s > 2$ it holds that:

$$\sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s} = \zeta(s)\zeta(s-1)$$

where ζ is the Riemann zeta function.

Proof. Let $F(z) = \sum_{m=1}^{\infty} \frac{f(m)}{m^z}$ and $G(z) = \sum_{n=1}^{\infty} \frac{g(n)}{n^z}$ be two complex series that converge absolutely somewhere in the complex plane then we define the convolution Dirichlet product as follows:

$$F(z)G(z) = \sum_{m=1}^{\infty} \frac{f(m)}{m^z} \sum_{n=1}^{\infty} \frac{g(n)}{n^z} = \sum_{n=1}^{\infty} \frac{h(n)}{n^z} = \sum_{n=1}^{\infty} \frac{(f * g)(n)}{n^z} = \sum_{n=1}^{\infty} \frac{\sum_{\mathrm{d}|\mathrm{n}} f(d) g\left(\frac{n}{d}\right)}{n^z}$$

So taking $f(n) = 1, \ g(n) = n$ we have that the convolution product is actually $\sigma(n)$ thus:

$$\sum_{m=1}^{\infty} \frac{1}{m^s} \sum_{n=1}^{\infty} \frac{n}{n^s} = \sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s} \Rightarrow \sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s} = \zeta(s)\zeta(s-1)$$

♦

Theorem 3

Let $\mathrm{lcm}(\cdot,\cdot)$ denote the least common multiple. For all $s > 1$ it holds that

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\mathrm{lcm}^s(m,n)} = \frac{\zeta^3(s)}{\zeta(2s)}$$

Proof. For all $M \in \mathbb{N}^+$ of the form $M = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ the number of solutions

$$\mathrm{lcm}(m,n) = M$$

is given by $(2a_1+1)(2a_2+1)\cdots(2a_k+1)$. It follows that the given series equals

$$\sum_{M=1}^{\infty}\frac{1}{M^s}\prod_{p|M}(2\nu_p(M)+1)$$

and since the function $\prod_{p|M}(2\nu_p(M)+1)$ is clearly a multiplicative one , it follows by Euler's product that

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\operatorname{lcm}^s(m,n)} &= \prod_{p\in\mathbb{P}}\left(1+\frac{3}{p^s}+\frac{5}{p^{2s}}+\frac{7}{p^{3s}}+\cdots\right)\\ &= \prod_{p\in\mathbb{P}}\frac{p^s\left(p^s+1\right)}{\left(p^s-1\right)^2}\\ &= \prod_{p\in\mathbb{P}}\frac{1-\frac{1}{p^{2s}}}{\left(1-\frac{1}{p^s}\right)^3}\\ &= \frac{\zeta^3(s)}{\zeta(2s)}\end{aligned}$$

♦

Author: Tolaso

JoM ... studies integrals

Theorem

Let $p : [a, b] \to \mathbb{R}$ be a polynomial such that $p(a) = 0$ and $\lim\limits_{x \to b^-} p(x) \csc x = \ell \in \mathbb{R}$. It holds that

$$\int_a^b p(x) \csc x \, \mathrm{d}x = 2 \sum_{n=0}^{\infty} \int_a^b p(x) \sin(2n+1)x \, \mathrm{d}x$$

Proof. We start from the well known identity

$$\sum_{n=0}^{N} \sin(2n+1)x = \frac{\csc x}{2} - \frac{\cos(2N+1)x}{2 \sin x}$$

Integrating we have

$$2 \sum_{n=0}^{N} \int_a^b p(x) \sin(2n+1)x \, \mathrm{d}x = \int_a^b p(x) \csc(x) \, \mathrm{d}x - \int_a^b p(x) \cdot \frac{\cos(2N+1)x}{\sin x} \, \mathrm{d}x$$

Now let $N \to +\infty$. Using the Riemann - Lebesgue Lemma we have that

$$\lim_{N \to +\infty} \int_a^b p(x) \cdot \frac{\cos(2N+1)x}{\sin x} \, \mathrm{d}x = 0$$

Hence,

$$\int_a^b p(x) \csc x \, \mathrm{d}x = 2 \sum_{n=0}^{\infty} \int_a^b p(x) \sin(2n+1)x \, \mathrm{d}x$$

♦

Other similar identities are:

(a) $\displaystyle\int_a^b p(x) \cot x \, \mathrm{d}x = 2 \sum_{n=1}^{\infty} \int_a^b p(x) \sin 2nx \, \mathrm{d}x$

(b) $\displaystyle\int_a^b p(x) \tan x \, \mathrm{d}x = -2 \sum_{n=1}^{\infty} (-1)^n \int_a^b p(x) \sin 2nx \, \mathrm{d}x$

Applications

1. It holds that

$$\int_0^{\pi/2} \theta^2 \cot\theta \, \mathrm{d}\theta = \frac{\pi^2}{4}\log 2 - \frac{7}{8}\zeta(3)$$

where ζ denotes the Riemann zeta function.

Proof. We're using a handy identity

$$\int_a^b p(x)\cot x \, \mathrm{d}x = 2\sum_{n=1}^{\infty}\int_a^b p(x)\sin 2nx \, \mathrm{d}x \tag{1}$$

Thus,

$$\begin{aligned}
\int_0^{\pi/2} \theta^2 \cot\theta \, \mathrm{d}\theta &= 2\sum_{n=1}^{\infty}\int_0^{\pi/2} \theta^2 \sin 2n\theta \, \mathrm{d}\theta \\
&= 2\sum_{n=1}^{\infty}\left(\frac{(2-\pi^2 n^2)(-1)^n - 2}{8n^3}\right) \\
&= \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3} + \frac{\pi^2}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^3} \\
&= -\frac{\eta(3)}{2} + \frac{\pi^2\ln 2}{4} - \frac{\zeta(3)}{2} \\
&= -\frac{3\zeta(3)}{8} + \frac{\pi^2\ln 2}{4} - \frac{\zeta(3)}{2} \\
&= \frac{\pi^2\ln 2}{4} - \frac{7\zeta(3)}{8}
\end{aligned}$$

♦

2. It holds that

$$\int_0^{\pi/2} \theta \csc\theta \, \mathrm{d}\theta = 2\mathcal{G}$$

where $\mathcal{G}$ denotes the Catalan's constant.

Proof. We're using a handy identity

$$\int_a^b p(x)\csc x \, \mathrm{d}x = 2\sum_{n=0}^{\infty}\int_a^b p(x)\sin(2n+1)x \, \mathrm{d}x \tag{1}$$

Thus,

$$\int_0^{\pi/2} \theta \csc\theta \, \mathrm{d}\theta = 2\sum_{n=0}^{\infty}\int_0^{\pi/2} \theta \sin\left(2n+1\right)\theta \, \mathrm{d}\theta$$

$$
\begin{aligned}
&= 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \\
&= 2\mathcal{G}
\end{aligned}
$$

♦

3. It holds that

$$\int_0^{\pi/4} \theta \tan\theta \, \mathrm{d}\theta = \frac{\mathcal{G}}{2} - \frac{\pi \ln 2}{8}$$

where $\mathcal{G}$ denotes the Catalan's constant.

Proof. We're using a handy identity

$$\int_a^b p(x) \tan x \, \mathrm{d}x = -2\sum_{n=1}^{\infty} (-1)^n \int_a^b p(x) \sin 2nx \, \mathrm{d}x \tag{1}$$

Thus,

$$
\begin{aligned}
\int_0^{\pi/4} x \tan x \, \mathrm{d}x &= -2\sum_{n=1}^{\infty} (-1)^n \int_0^{\pi/4} x \sin 2nx \, \mathrm{d}x \\
&= 2\sum_{n=1}^{\infty} (-1)^n \left(\frac{\pi n \cos \frac{n\pi}{2} - 2 \sin \frac{n\pi}{2}}{8n^2} \right) \\
&= 2\sum_{n=1}^{\infty} (-1)^n \left(\frac{\pi \cos \frac{n\pi}{2}}{8n} - \frac{\sin \frac{n\pi}{2}}{4n^2} \right) \\
&= \frac{\pi}{4} \sum_{n=1}^{\infty} \frac{(-1)^n \cos \frac{n\pi}{2}}{n} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n \sin \frac{n\pi}{2}}{n^2} \\
&= \frac{\mathcal{G}}{2} - \frac{\pi \ln 2}{8}
\end{aligned}
$$

♦

Suggested Problems

1. Given a convergent series $\sum_{n=1}^{\infty} a_n$ prove that the Dirichlet series $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ converges uniformly in the upper half-plane $0 \leq s < +\infty$. Using this, prove that

$$\lim_{s \to 0^+} \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \sum_{n=1}^{\infty} a_n$$

2. Let μ denote the Möbius function. Prove that

$$\lim_{s \to 1^+} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = 0$$

3. Let μ denote the Möbius function. Prove that

$$\lim_{N \to +\infty} \sum_{n=1}^{N} \frac{|\mu(n)|}{n} = +\infty$$

4. Let μ denote the Möbius function and $\psi^{(0)}$ the digamma function. Prove that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \psi^{(0)}\left(1 + \frac{1}{n}\right) = \frac{1}{2}$$

5. Let μ denote the Möbius function. For $\mathfrak{Re}(s) > 1$ evaluate the series

$$\mathcal{S} = \sum_{n=1}^{\infty} \frac{(-1)^{\mu(n)}}{n^s}$$

6. Evaluate the integral

$$\mathcal{J} = \int_0^{\pi/2} x^7 \cot x \, \mathrm{d}x$$

Conclude the value of $\int_0^{\pi/2} x^6 \ln \sin x \, \mathrm{d}x$.