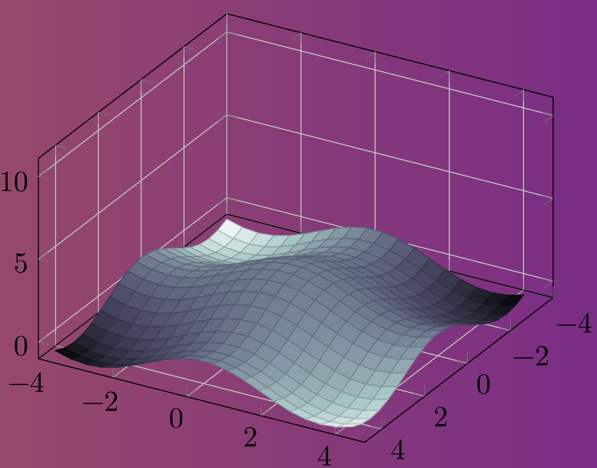
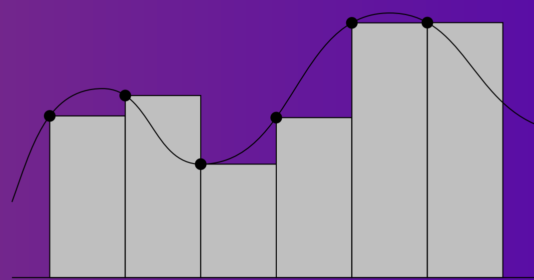
Mathematical Analysis

A collection of problems

Real & Complex Analysis • General Topology • Multivariable Calculus • Integrals and Series





Version 6

Tolaso J. Kos

Mathematical Analysis

A collection of problems

Title of Project	A collection of Problems in ... Analysis
Language	(American) English
Editor:	Tolaso J. Kos
Website:	tolaso.com.gr
Network:	Tolaso Network

Typesetting:	LaTeX (including TeXLive 2016 + PDF LaTeX)
Version:	v.1.6 First version: March 26, 2017 Second version: May 3, 2017 Third version: May 12, 2017 Fourth version: July 27, 2017 Fifth version: August 30, 2017 Sixth version: November 18, 2017

Social Media and Email Account

Joy of Mathematics

Joy of Mathematics

tolaso@tolaso.com.gr

Tolaso J Kos

Tolaso 94

Mini Contents

Background

There are a lot of interesting analysis problems scattered in the Internet World. Navigating through different sites you may encounter an exercise that will catch your attention and possibly you may want to archive it in your collection so to have access to it later.

This is the main idea behind this booklet. The attempt started back in 2014 when an effort to collect as many exercises as possible began. Basic ideas are being recycled frequently and reappear in many exercises although unrelated at first.

The author (Tolaso) started the collection of the problems using exercises that he encountered in his university classes (Calculus I, Calculus II and Calculus IV) and found to be the most interesting and fascinating. He dediced to include non trivial problems (as these have nothing to offer usually and rely mostly on definitions) but challenging ones.

The first collection contained 125 problems along with their solutions. That project was in Greek and when it was translated in English the solution section was frozen and was never translated. Now that the collection expands the author is trying his very best to keep up with the typesetting of the solutions' section. Hopefully , it will be completed one day.

Foreword

Dear reader,

the following booklet contains a collection of interesting problems in Mathematical Analysis. The problems come from various branches of mathematics.

- Real and Complex Analysis
- Multivariable Calculus
- General Topology
- Integrals and Series

In each section the reader of this booklet shall encounter exercises that may find out there. Many of them are known to you but still they are interesting. However , there do exist exercises that demand creativity in order to be solved. The level of difficulty varies from exercise to exercise and in no way are the problems ordered according to their level of difficulty.

The version you are now reading is Version 6 which is an improvement of the previous Golden Version 5. This new version has seen the correction of some typos in the previous version as well as the introduction of the Mini Contents section which has clickable links to the sections this booklet contains.

I mentioned earlier that a lot of typos in the exercises were corrected and this is because of all the readers. I would like to personally thank all those people who contacted me personally to mention any typographical errors and / or mathematical errors. A big thanks to all of you guys !

I would like to remind you (but you already know the deal) that I am open to your e-mails to improvements / suggestions . Feel free to contact me at the e-mail address that you will find in page 2. Last but not least , you are free to use the booklet as an instructive tutorial to your students. However , be very careful when assigning exercises to them.

Tolaso J Kos

November 18, 2017

Acknowledgements

- Many thanks to all those people (from all around the world) who embraced this booklet and have sent remarks and / or suggestions so that it is improved as well as selecting some of its exercises to assign to their students. I really appreciate it.
- The people at TeX Stack Exchange who have suggested some hacks for some parts of the existing code so that everything fits within the specified margins.

Real - Complex Analysis

1. For which $a \in \mathbb{R}$ does the sequence

$$\gamma_n = (1+a)(1+2a^2)\cdots(1+na^n)$$

converge? Give a brief explanation.

2. A sequence of real number $\{x_n\}_{n\in\mathbb{N}}$ satisfies the condition

$$|x_n - x_m| > \frac{1}{n} \quad \text{whenever} \quad n < m$$

Prove that x_n is not bounded.

3. Prove that

$$\lim_{n\to+\infty}\left((n+1)^{(n+2)/(n+1)} - n^{(n+1)/n}\right) = 1$$

4. Find the value of

$$\ell = \sqrt{6+\sqrt{6+\sqrt{6+\sqrt{\cdots}}}}$$

5. Let $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$ and $\{y_n\}_{n=1}^{\infty} \subset (0,+\infty)$. Suppose that $\{x_n/y_n\}_{n=1}^{\infty}$ is monotone. Prove that the sequence $\{z_n\}_{n\in\mathbb{N}}$ defined as

$$z_n = \frac{x_1+x_2+\cdots+x_n}{y_1+y_2+\cdots+y_n}$$

is also monotone.

6. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence defined as

$$x_n = \sin 1 + \sin 3 + \sin 5 + \cdots + \sin(2n-1)$$

Find the supremum as well as the infimum of the sequence x_n.

7. Let $\alpha \in \mathbb{R}$ such that $\alpha/\pi \notin \mathbb{Q}$. Prove that the sequence

$$\omega_n = \sin(\sin\alpha) + \sin(\sin(2\alpha)) + \cdots + \sin(\sin(n\alpha))$$

is bounded.

8. Let $\{a_n\}_{n\in\mathbb{N}}$ be a real valued sequence such that the series $\sum\limits_{n=1}^{\infty} a_n^2$ converges. Prove that the series $\sum\limits_{n=1}^{\infty} \frac{a_n}{n}$ also converges.

9. Let $\{a_n\}_{n\in\mathbb{N}}$ be a positive real valued sequence. If the series $\sum\limits_{n=1}^{\infty} a_n$ converges prove that the series $\sum\limits_{n=1}^{\infty} a_n^{n/(n+1)}$ also converges.

10. Let $\alpha \in \mathbb{R}\setminus\mathbb{Z}$ and let us denote with $\lfloor\cdot\rfloor$ the floor function. Prove that the series

$$\mathcal{S} = \sum_{n=1}^{\infty}\left(\alpha - \frac{\lfloor n\alpha\rfloor}{n}\right)$$

diverges.

(16th Cuban Mathematical Olympiad)

11. Let a_n be a positive and strictly decreasing sequence such that $\lim a_n = 0$. Prove that the series

$$\mathcal{S} = \sum_{n=1}^{\infty}\frac{a_n - a_{n+1}}{a_n}$$

diverges. 🛢

12. Let $\mathbb{P}$ denote the set of prime numnbers. Discuss the convergence of the series

$$\mathcal{S} = \sum_{p\in\mathbb{P}}\frac{\sin p}{p}$$

13. Examine whether the (double) series

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\sin(\sin(nm))}{n^2+m^2}$$

converges. 🛢

🛢***Hint:*** Let $x_1,\ldots,x_n \in (0,1)$. It holds that

$$\sum_{i=1}^{n}(1-x_i) \geqslant 1 - \prod_{i=1}^{n}x_i$$

🛢It appears that this problem is quite difficult. It appeared in several fora including math.stackexchange.com as well as mathematica.gr. In both went answered till today. In math.stackexchange.com they suggest that the series converges and its limit is $\frac{1}{2}$.

14. Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of strictly increasing positive integers. For each $n \geqslant 1$ let W_n be the least common multiple of the first n terms $X_1, X_2, \ldots, X_n$. Prove that , as $n \to +\infty$, the series

$$\mathcal{S} = \frac{1}{W_1} + \frac{1}{W_2} + \cdots + \frac{1}{W_n}$$

converges.

15. Let $\{a_n\}_{n\in\mathbb{N}}$ be a strictly increasing sequence of positive integers. Prove that the series $\displaystyle\sum_{i=0}^{n} \frac{1}{[a_i, a_{i+1}]}$ converges. Here $[\cdot,\cdot]$ denotes the least common multiple.

16. Let $f : (0, +\infty) \to \mathbb{R}$ be a positive differentiable function such that its derivative is positive. Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{f(n)}$ converges if-f the series $\displaystyle\sum_{n=1}^{\infty} \frac{f^{-1}(n)}{n^2}$ converges.

17. Let $\mathcal{H}_n$ denote the n-th harmonic number. Study the convergence of the series

$$\mathcal{S} = \sum_{n=1}^{\infty} \alpha^{\mathcal{H}_n}$$

for the different values of $\alpha > 0$.

18. Let $\mathcal{H}_n$ denote the n -th harmonic number. Prove that the series

$$\mathcal{S} = \sum_{n=1}^{\infty} (-1)^n \frac{\sqrt[n]{\log n!}}{\log\left(\mathcal{H}_{n+1}\right)}$$

converges.

Hint:

$$\begin{aligned}
\sum_{i=0}^{n} \frac{1}{[a_i, a_{i+1}]} &= \sum_{i=0}^{n} \frac{(a_i, a_{i+1})}{a_i a_{i+1}} \\
&\leqslant \sum_{i=0}^{n} \frac{a_{i+1} - a_i}{a_i a_{i+1}} \\
&= \sum_{i=0}^{n} \frac{1}{a_i} - \frac{1}{a_{i+1}} \\
&= \frac{1}{a_0} - \frac{1}{a_n} < \frac{1}{a_0}
\end{aligned}$$

19. Let $\{a_n\}_{n\in\mathbb{N}}$ be a positive real valued sequence such that the series $\sum\limits_{n=1}^{\infty} a_n$ converges. Examine the convergence of the series

$$\mathcal{S} = \sum_{n=1}^{\infty} \left(1 - \frac{\sin a_n}{a_n}\right)$$

20. Let $\{x_n\}_{n\in\mathbb{N}}$ be a real valued sequence of positive terms such that $\sum\limits_{n=1}^{\infty} x_n$ converges. Set

$$s_n = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}$$

Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{x_n s_n^2}$ converges.

21. Let $\alpha \in \mathbb{R}$. For which values of α does the series

$$\mathcal{S} = \sum_{n=1}^{\infty} \left(\frac{\pi}{2} - \arcsin \frac{n}{n+4}\right)^{\alpha}$$

converge?

22. Examine the convergence of the series

$$\mathcal{S} = \sum_{n=1}^{\infty} \frac{\sin(\sin n)}{n}$$

Does it converge absolutely? Justify your answer.

23. For what values of $x \in \mathbb{R}$ do the series

$$\text{(i)}\ \ \mathcal{S}_1 = \sum_{n=1}^{\infty} \cos(2^n x) \qquad \text{(ii)}\ \ \mathcal{S}_2 = \sum_{n=1}^{\infty} \sin(2^n x)$$

converge?

24. What is the monotony of the function

$$f(j) = \prod_{i=-j}^{0} \sum_{k=0}^{\infty} \frac{i^k}{k!}, \quad j \in \mathbb{Z}$$

25. Let $f : [-\pi, \pi] \to \mathbb{R}$ be a Riemann integrable function. Prove that

$$\lim_{n\to+\infty} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \lim_{n\to+\infty} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = 0$$

26. Prove , without using special functions, that the integral $\displaystyle\int_0^{\pi} \frac{\ln x}{x+\pi} \, dx$ converges.

27. Let $f_n(x) : [0,1] \to \mathbb{R}$ be a sequence of functions converging uniformly to a function f. Prove that

$$\lim_{n\to+\infty} \int_{1/n}^{1} f_n(x)\,dx = \int_0^1 f(x)\,dx$$

28. What can you say about the uniform convergence of the series

$$\mathcal{S} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\pi x)\,, \quad x \in \mathbb{R}$$

29. Let $f, g : \mathbb{R} \to \mathbb{R}$ be 1 periodic and continuous functions. Prove that

$$\lim_{n\to+\infty} \int_0^1 f(x)g(nx)\,dx = \int_0^1 f(x)\,dx \int_0^1 g(x)\,dx$$

30. Let $f : (0,+\infty) \to \mathbb{R}$ such that , forall $x > 0$, the limit $\lim\limits_{n\to+\infty} f(nx) \in \mathbb{R}$. Examine if the limit $\lim\limits_{x\to+\infty} f(x)$ exists in $\mathbb{R}$ if:

(a) f is a continuous function.

(b) f is an arbitrary function.

31. (a) Give an example of a bounded function $f : (0,+\infty) \to \mathbb{R}$ such that the limit $\ell = \lim\limits_{x\to 0^+} f(x)$ does not exist.

(b) If f is a function such as described in (a) then examine if the following limits exist.

(i)

$$\ell_1 = \lim_{x\to 0^+} x f(x)$$

(ii)

$$\ell_2 = \lim_{x\to 0^+} (1-x) f(x)$$

32. Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Prove that

$$\lim_{n\to+\infty} \int_a^b \frac{f(x)}{3 + 2\cos nx}\,dx = \frac{1}{\sqrt{5}} \int_a^b f(x)\,dx$$

33. Prove that

$$\min_{a_i\in\mathbb{R}} \int_0^1 \left| x^n + a_1 x^{n-1} + \cdots + a_n \right| dx = \frac{1}{4^n}$$

34. Let $x \in \mathbb{R}$. Consider the series

$$\mathcal{S} = \sum_{n=2}^{\infty} \frac{\sin nx}{\log n} \tag{1}$$

(A) (a) Prove that $\mathcal{S}$ converges forall $x \in \mathbb{R}$.

(b) Prove that (1) is not a Fourier series of a Lebesgue integrable function.

(B) Examine if the function defined at (1) is continuous. Give a brief explanation to support your argument.

(C) Prove that the series $\displaystyle\sum_{n=2}^{\infty} \frac{\cos nx}{\log n}$ is both Riemann and Lebesgue integrable as well as a Fourier series.

35. Let $a \in \mathbb{Z}$. Define the function

$$f(x) = \sin ax,\ x \in (0,\pi)$$

Prove that f can be expanded into a Fourier cosine series and that it holds

$$\sin ax \sim \begin{cases} \dfrac{4a}{\pi} \displaystyle\sum_{n=0}^{\infty} \frac{\cos(2n+1)x}{a^2 - (2n+1)^2} & , \quad a \text{ even} \\ \dfrac{4a}{\pi} \left[\dfrac{1}{2a^2} + \displaystyle\sum_{n=1}^{\infty} \frac{\cos 2nx}{a^2 - 4n^2} \right] & , \quad a \text{ odd} \end{cases}$$

36. Let p, q be two points and γ be a curve passing through these two points. Prove that

(a) $\gamma'(t) \cdot u \leqslant \|\gamma'(t)\|$ where u is an arbitrary unit vector.

(b) that the segment of the curve γ between the points p and q has length at least equal to the distance $\|q - p\|$ by considering as $u = \frac{q-p}{\|q-p\|}$.

***Hint*:** It holds that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin n\pi x = \begin{cases} \dfrac{\pi x}{2} & , \quad 0 \leqslant x < 1 \\ 0 & , \quad x = 1 \\ \dfrac{\pi(x-2)}{2} & , \quad 1 < x \leqslant 2 \end{cases}$$

Do the same question for the quite similar series $\displaystyle\sum_{n=2}^{\infty} \frac{\sin nx}{n \log n}$.

The conclusion of this exercise is to show that the line is the shortest distance between two points.

37. Let $\{a_n\}_{n\in\mathbb{N}}$ be a bounded sequence. Prove that the sequence of functions defined as $\sum_{n=1}^{\infty} \frac{a_n}{n^{2x}}$ converges absolutely and uniformly on $(0, +\infty)$ to a differentiable function.

(Question from a Real Analysis Exam University of Ioannina , Greece)

38. Let $f : [-\pi, \pi] \to \mathbb{R}$ be defined as $f(x) = |x|$.

(a) Expand f in a Fourier series.

(b) Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \qquad \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

(c) Apply Parseval's identity to evaluate the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

39. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Prove that there exist functions g_i, $i = 1, \ldots, n$ such that

$$f(x_1, x_2, \ldots, x_n) - f(0, 0, \ldots, 0) = \sum_{i=1}^{n} x_i g_i(x_1, x_2, \ldots, x_n)$$

40. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f'(x) = 0$ forall $x \in \mathbb{Q}$. Does it necessarily follow that f is constant throughout $\mathbb{R}$? Explain your answer.

41. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that preserve convergent series. *(That is a function preserves convergent series in the sense mentioned above if* $\sum f(a_n)$ *converges whenever* $\sum a_n$ *converges.)* [⛁]

42. Examine if there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x)) = x^2 + 1 \text{ forall } x \in \mathbb{R}$$

43. Examine if there exists an $1-1$ function $f : \mathbb{N} \to \mathbb{N}$ such that $\sum_{n=1}^{\infty} \frac{f(n)}{n^2}$ converges.

[⛁] The answer to this difficult question is that the only functions with this property are of the form $f(x) = \lambda x,\ x \in (-\delta, \delta)$.

44. Examine whether the series

$$\mathcal{S} = \sum_{n=1}^{\infty} \sin\left[\pi\left(2+\sqrt{3}\right)^n\right]$$

converges.

45. Examine whether the series

$$\mathcal{S} = \sum_{n=1}^{\infty} \left(e - (1+\tfrac{1}{n})^n\right)$$

converges.

46. Given a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that forall $x \in \mathbb{R} \setminus \{0, 1\}$

$$\int_0^x f(t)\,dt > \int_x^1 f(t)\,dt \tag{1}$$

prove that $\int_0^1 f(t)\,dt = 0$.

47. Let $f : [a, b] \to \mathbb{R}$ be a continuously differentiable function such that $f(a) = f(b) = 0$ and $\int_a^b f^2(t)\,dt = 1$. Prove that:

(a) $\displaystyle\int_a^b x f(x) f'(x)\,dx = -\frac{1}{2}$

(b) $\displaystyle\int_a^b \left(f'(x)\right)^2 dx \int_a^b x^2 f^2(x)\,dx > \frac{1}{4}$

48. Let

$$f(x) = \sin x \sin(2x) \sin(4x) \cdots \sin(2^n x)$$

Prove that

$$|f(x)| \leqslant \frac{2}{\sqrt{3}} \left| f\left(\frac{\pi}{3}\right) \right|$$

49. Prove that for every $x \in \mathbb{R}$ the inequality

$$\frac{x^{2n}}{(2n)!} + \frac{x^{2n-1}}{(2n-1)!} + \cdots + \frac{x^2}{2!} + x + 1 > 0$$

holds.

50. Prove that for arbitrary real numbers $a_1, a_2, \ldots, a_n$ the following inequality holds.

$$\sum_{m,n=1}^{k} \frac{a_m a_n}{m+n} \geqslant 0$$

51. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of positive real numbers. Prove that

$$\limsup_{n\to+\infty}\left(\frac{a_1+a_{n+1}}{a_n}\right)^n \geqslant e$$

52. Let $\mathcal{C}$ denote the Cantor set. We define the function $\chi_{\mathcal{C}}:[0,1]\to\mathbb{R}$ as follows:

$$\chi_{\mathcal{C}}=\begin{cases}1 & , \quad x\in\mathcal{C}\\ 0 & , \quad \text{elsewhere}\end{cases}$$

(a) Prove that $\chi_{\mathcal{C}}$ is Riemann integrable.

(b) Evaluate $\int_0^1 \chi_{\mathcal{C}}(x)\,dx$.

53. Let $\{a_n\}_{n\in\mathbb{N}}$ be a real valued sequence such that the series $\sum\limits_{n=1}^{\infty}\frac{a_n}{n}$ converges. Prove that

$$\lim_{n\to+\infty}\frac{a_1+a_2+\cdots+a_n}{n}=0$$

54. Prove that the function $f:\mathbb{R}^n\setminus\{0\}\to\mathbb{R}$ defined as

$$f(\mathbf{x})=\frac{\mathbf{x}}{\|\mathbf{x}\|^a}\ ,\ a>0$$

is a vector field but its domain is not star-shaped.

55. Does the ordered field of the rational functions satisfy the axiom of completeness? Explain your answer.

A solution goes along these lines:

$$\begin{aligned}\sum_{m,n=1}^{k}\frac{a_m a_n}{m+n}&=\sum_{m,n=1}^{k}\int_0^1 a_m a_n t^{m+n-1}\,dt\\&=\int_0^1\left(\sum_{m,n=1}^{k}a_m a_n t^{m+n-1}\right)dt\\&=\int_0^1\left(\sum_{m=1}^{k}a_m t^{m-1/2}\right)^2 dt\\&\geqslant 0\end{aligned}$$

In fact the above inequality tells us that the matrix $\left[\frac{1}{m+n}\right]_{m,n=1}^{k}$ is positive semidefinite.

This is a very difficult exercise. One solution may be found at M. Hata's notes. Another solution is to contradict the result and move along those lines.

56. Let $f:[2,+\infty)\to\mathbb{R}$ be a uniformly continuous function. Prove that the integral

$$\mathcal{J}=\int_2^{\infty}\frac{f(x)}{x^2\log^2 x}\,dx$$

converges.

57. Let $f:[a,b]\to\mathbb{R}$ be a Riemann integrable function. If $f(x)=0$ forall rationals of the interval $[a,b]$ then prove that $\int_a^b f(x)\,dx=0$.

58. Prove that there exists no rational function such that

$$f(n)=1+\frac{1}{2}+\cdots+\frac{1}{n}$$

forall $n\in\mathbb{N}$.

59. Let $f:\mathbb{R}\to(0,+\infty)$ be a function such that forall $x\in\mathbb{R}$ it holds that

$$f(x)\log f(x)=e^x \tag{1}$$

Evaluate the limit

$$\ell=\lim_{x\to+\infty}\left(1+\frac{\log x}{f(x)}\right)^{f(x)/x}$$

(Romania , 1986)

60. Let $n\in\mathbb{N}$ and let $f:[-1,1]\to\mathbb{R}$ be a continuous function such that

$$\int_{-1}^{1}x^{2n}f(x)\,dx=0$$

Prove that f is odd.

61. Let $\mathcal{G}$ denote the Catalan constant. Prove that

$$\log\left(1+\sqrt{2}\right)<\int_0^1\frac{\tanh x}{x}\,dx<\mathcal{G}$$

62. Evaluate the limit

$$\Omega=\lim_{n\to+\infty}\sum_{k=1}^{n}\frac{\frac{1}{n}\arctan\left(\frac{k}{n}\right)}{1+2\sqrt{1+\frac{1}{n}\arctan\left(\frac{k}{n}\right)}}$$

(Dan Sitaru)

63. Evaluate the limit

$$\Omega = \lim_{n\to+\infty} \sum_{k=1}^{n} \arcsin \frac{1}{\sqrt{n^2+k}}$$

64. Let φ denote Euler's totient function. Evaluate the limit

$$\ell = \lim_{n\to+\infty} \frac{1}{n^2} \sum_{k=1}^{n} \sin\left(\frac{\pi k}{n}\right) \varphi(k)$$

65. Let $\alpha > 0$. Prove that:

$$\lim_{n\to+\infty} \frac{1}{\log n} \sum_{1\leqslant k\leqslant n^a} \frac{1}{k}\left(1-\frac{1}{n}\right)^k = \min\{1, a\}$$

66. Let us denote with ζ the Riemann zeta function with $\zeta(0) = -\frac{1}{2}$. Let us also denote with $\zeta^{(n)}$ the n – th derivative of zeta. Evaluate the limit

$$\ell = \lim_{n\to+\infty} \frac{\zeta^{(n)}(0)}{n!}$$

⛃

67. Let ζ denote the Riemann zeta function. Evaluate the limit

$$\ell = \lim_{n\to+\infty} n\left(\zeta(2) - \sum_{k=1}^{n} \frac{1}{k^2}\right)$$

68. Let Γ denote the Euler's Gamma function. Prove that

$$\frac{\Gamma\left(\frac{1}{10}\right)}{\Gamma\left(\frac{2}{15}\right)\Gamma\left(\frac{7}{15}\right)} = \frac{\sqrt{5}+1}{3^{1/10}2^{6/5}\sqrt{\pi}}$$

69. Consider the real valued sequence $\{y_n\}_{n\in\mathbb{N}}$ such that forall real valued sequences $\{x_n\}_{n\in\mathbb{N}}$ with $\lim x_n = 0$ the series $\sum\limits_{n=1}^{\infty} x_n y_n$ converges. Prove that the series $\sum\limits_{n=1}^{\infty} |y_n|$ also converges.

70. Let $\{a_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of positive terms. Prove that the series $\sum a_n \sin nx$ converges uniformly throughout $\mathbb{R}$ if and only if $na_n \to 0$.

⛃The above limit tells us that $\zeta^{(n)}(0) \sim -n!$.

71. Let $\{a_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of positive terms. Prove that the series $\sum\limits_{n=1}^{\infty} a_n \cos nx$ converges uniformly on $\mathbb{R}$ if and only if the series $\sum\limits_{n=1}^{\infty} a_n$ converges.

72. Given the sequence of functions

$$f_n(x) = \cos^n x,\ 0 \leqslant x \leqslant \pi$$

Prove that

(a) $\lim f_n(x) = 0$ but $f_n(\pi)$ does not converge.

(b) Prove that f_n converges pointwise but not uniformly on $[0, \pi/2]$.

73. Let $f : [0, +\infty) \to \mathbb{R}$ be an integrable and uniformly continuous function. Prove that $\lim\limits_{x\to+\infty} f(x) = 0$. Does this result hold if we drop the assumption of the *uniformly continuous* ? Explain your answer.

74. Define a function $f : \mathbb{R} \to \mathbb{R}$ such that for every $q \in \mathbb{Q}$ must hold $f(q) \in \mathbb{Q}$ but $f'(q) \notin \mathbb{Q}$.

75. Given the sequence of $f_n : \mathbb{R} \to \mathbb{R}$ where $n \in \mathbb{N}$ defined as

$$f_n(x) = \sum_{n=1}^{\infty} \frac{n}{n^3 + x^2}$$

prove that

(a) the serieses $\sum\limits_{n=1}^{\infty} f_n$ and $\sum\limits_{n=1}^{\infty} f'_n$ converge uniformly to functions $f, g : \mathbb{R} \to \mathbb{R}$.

(b) the functions f, g are continuous.

(c) $f' = g$.

(d) it holds that

(i) $\displaystyle\int_{-1}^{1} f(x)\,dx = 2\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \arctan \frac{1}{n\sqrt{n}}$ ⛃

(ii) $\displaystyle\int_{-\pi}^{\pi} x^4 g(x)\,dx = 0.$

76. Let $f : [0, 1] \to \mathbb{R}$ be defined as:

$$f(x) = \begin{cases} 0 & , \quad x \in [0,1] \cap (\mathbb{R} \setminus \mathbb{Q}) \\ x_n & , \quad x = q_n \in [0,1] \cap \mathbb{Q} \end{cases}$$

⛃What can you say about the integral $\int_{-\infty}^{\infty} f(t)\,dt$? Does it converge?

where x_n is a sequence such that $\lim x_n = 0$ and $0 \leqslant x_n \leqslant 1$ and q_n be an enumeration of the rationals of the interval $[0,1]$. Prove that f is Riemann integrable and that $\int_0^1 f(x)\,dx = 0$.

77. Let f be holomorphic on the open unit disk $\mathbb{D}$ and suppose that

$$\iint_{\mathbb{D}} |f(z)|^2 \, d(x,y) < +\infty$$

If the Taylor expansion of f is of the form $\sum\limits_{n=0}^{\infty} a_n z^n$ then prove that the series $\displaystyle\sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1}$ converges.

78. Let f_n be a sequence of real valued $\mathcal{C}^1$ functions on $[0,1]$ such that forall $n \in \mathbb{N}$ the following hold:

- $|f_n'(x)| \leqslant \dfrac{1}{\sqrt{x}} \quad (0 < x \leqslant 1)$
- $\int_0^1 f_n(x)\,dx = 0$

Prove that f_n has a convergent subsequence that converges uniformly on $[0,1]$.

79. Let $\chi_{\mathbb{Q}}$ denote the characteristic function of the rationals in $[0,1]$. Does there exist a sequence of continuous functions $f_n : [0,1] \to \mathbb{R}$ such that f_n converges to $\chi_{\mathbb{Q}}$ pointwise?

80. Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that

$$\int_0^1 f(t)\,dt = \int_0^1 t f(t)\,dt = 1 \tag{1}$$

Prove that $\int_0^1 f^2(t)\,dt \geqslant 4$.

81. Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that

$$\int_0^1 f(t)\,dt = \int_0^1 t f(t)\,dt \tag{1}$$

Prove that there exists a $c \in (0,1)$ such that

$$\int_0^c f(t)\,dt = \frac{c}{2} \int_0^c f(t)\,dt$$

82. Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that

$$\int_0^1 f(t)\,dt = \int_0^1 t f(t)\,dt \tag{1}$$

Prove that there exists a $c \in (0,1)$ such that

$$c f(c) = 2 \int_c^0 f(t)\,dt$$

83. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that

$$f'(x) = f^2(x) f(-x) \tag{1}$$

Find an explicit formula for f.

84. Let $f : [0,1] \to \mathbb{R}$ be a continous function such that $\int_0^1 f(t)\,dt = 1$ and

$$\int_0^1 (1 - f(x))\, e^{-f(x)}\,dx \leqslant 0 \tag{1}$$

Prove that $f(x) = 1$ forall $x \in \mathbb{R}$.

85. Let $f : [a,b] \to [0,+\infty)$ be a continous and not everywhere 0 function. Prove that

$$\lim_{n \to +\infty} \frac{\int_a^b f^{n+1}(t)\,dt}{\int_a^b f^n(t)\,dt} = \sup_{x \in [a,b]} f(x)$$

86. Prove that

$$\lim_{n \to +\infty} n \sin(2\pi e n!) = 2\pi$$

87. Prove that the limit

$$\ell = \lim_{n \to +\infty} \frac{\tan n}{n}$$

does not exist.

88. Examine if there exists a continuous function $f : [1,+\infty) \to \mathbb{R}$ such that $f(x) > 0$ forall $x \in [1,+\infty)$ and $\int_1^{\infty} f(t)\,dt$ converges whereas $\int_1^{\infty} f^2(t)\,dt$ diverges. [*]

89. Let $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and consider the function

$$f(x) = a_1 \tan x + a_2 \tan \frac{x}{2} + \cdots + a_n \tan \frac{x}{n}$$

where $a_1, a_2, \ldots, a_n \in \mathbb{R}$ and $n \in \mathbb{N}$. If $|f(x)| \leqslant |\tan x|$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ then prove that

$$\left| a_1 + \frac{a_2}{2} + \cdots + \frac{a_n}{n} \right| \leqslant 1$$

[*] Do the same exercise with the extra assumption that f is uniformly continuous.

90. Let $f:[0,1]\to\mathbb{R}$ be a twice differentiable function with a continuous second derivative. If n is a natural number greater than 1 such that

$$\sum_{k=1}^{n-1} f\left(\frac{k}{n}\right) = -\frac{f(0)+f(1)}{2}$$

then prove that

$$\left(\int_0^1 f(t)\,dt\right)^2 \leqslant \frac{1}{5!n^4}\int_0^1 \left(f''(t)\right)^2\,dt$$

91. Prove that every function $f:\mathbb{Q}\to\mathbb{Q}$ can be written as the sum of two $1-1$ functions $g,h:\mathbb{Q}\to\mathbb{Q}$.

92. Give an example of a function $f:\mathbb{R}\to\mathbb{R}$ such that any rational number is its period but any irrational is not. Also, prove that there exists no function $g:\mathbb{R}\to\mathbb{R}$ such that any irrational is its period and any rational is not.

93. Let $\mathcal{H}_n$ denote the n -th Harmonic number. Prove the inequality

$$\frac{\pi^2}{6}\left(\zeta(3)-\frac{\pi^2}{12}\right) < \sum_{n=1}^{\infty}\frac{e^{\mathcal{H}_n}\log\mathcal{H}_n}{n^3}$$

[1]

94. Prove that for an entire function f holding $\lim\limits_{z\to\infty}\frac{f(z)}{z}=0$ then f is constant.

95. Let $f:\mathbb{C}\to\mathbb{C}$ be an analytic and $1-1$ function and let $\mathbb{D}$ be the open unit disk. Prove that

$$\iint\limits_{\mathbb{D}} \left|f'(z)\right|\,dz = \operatorname{area}(f(\mathbb{D}))$$

[2]

96. Let $n\in\mathbb{N}$ and f be an entire function. Prove that for any arbitrary positive numbers a,b it holds that

$$\frac{\int_0^{2\pi} e^{-int} f\left(z+ae^{it}\right)\,dt}{\int_0^{2\pi} e^{-int} f\left(z+be^{it}\right)\,dt} = \left(\frac{a}{b}\right)^n$$

97. Let $a,b\in\mathbb{C}$ such that $|b|<1$. Prove that

$$\frac{1}{2\pi}\oint\limits_{|z|=1}\left|\frac{z-a}{z-b}\right|^2|dz| = \frac{|a-b|^2}{1-|b|}+1$$

[1] You might consider ideas from this link.
[2] This is known as Lusin Area Integral Formula.

98. Define

$$f(z)=\frac{1}{z}\cdot\frac{1-2z}{z-2}\cdots\frac{1-10z}{z-10}$$

Evaluate the contour integral $\displaystyle\oint\limits_{|z|=100} f(z)\,dz$.

99. Prove that there does not exist a sequence $\{p_n(z)\}_{n\in\mathbb{N}}$ of complex polynomials such that $p_n(z)\to\frac{1}{z}$ uniformly on $\mathcal{C}_R=\{z\in\mathbb{C}\mid |z|=R\}$.

100. Let f be a meromorphic function on a (connected) Riemann Surface X. Show that the zeros and the poles of f are isolated points.

101. Let us prove that $0=1$. We begin by stating Picard's Little Theorem:

Theorem

If a function $f:\mathbb{C}\to\mathbb{C}$ is entire and non-constant, then the set of values that $f(z)$ assumes is either the whole complex plane or the plane minus a single point.

Let us now consider $g(z)=e^x$ which is definitely complex differentiable. Since the composition of complex differentiable functions is also complex differentiable then the function

$$f(z)=g\left(g(x)\right)=e^{e^z}$$

is also complex differentiable. Also, f is not constant; that is for sure. Since there exists no z such that $e^z=0$ then 0 and 1 are not in the range of f. However, this is an obscurity unless $0=1$.

Find the flaw in the above argument. [3]

102. Let $A\subseteq\mathbb{R}$ be a set of finite measure.

(a) Find the Fourier series of $|\sin\lambda x|$.

(b) Evaluate the limit

$$\ell = \lim_{\lambda\to+\infty}\int\limits_A|\sin\lambda x|\,dx$$

103. Let $\langle\cdot,\cdot\rangle$ denote the usual inner product of $\mathbb{R}^m$. Evaluate the integral

$$\mathcal{M}=\int\limits_{\mathbb{R}^m}\exp\left(-(\langle x,\mathcal{S}^{-1}x\rangle)^a\right)\,dx$$

[3] The flaw is not in the theorem!

where $\mathcal{S}$ is a positive symmetric $m \times m$ matrix and $a > 0$.[1]

104. Let $\psi^{(n)}$ denote the n - th polygamma function and let $n \in \mathbb{N} \cup \{0\}$. Prove that

$$\frac{\psi^{(n)}(z)}{\psi^{(n+1)}(z)} \geqslant \frac{\psi^{(n+1)}(z)}{\psi^{(n+2)}(z)} \quad , \quad z > 0$$

[2]

105. Consider the points $O(0,0)$ and $A(1,0)$. Let $\Gamma(x,y)$ be a point of the plane such that $y > 0$. Set $\varphi(x,y)$ to be the angle that is defined by $O\Gamma$ and $A\Gamma$. (the one that is less than π.) Prove that the function $\varphi(x,y)$ is harmonic.

Multivariable Calculus

106. Given the curve $\gamma(t) = e^{-t}(\cos t, \sin t)$, $t \geqslant 0$

(a) Sketch its graph.

(b) Evaluate the length of the curve as well as the following line integrals

$$\text{(i)} \oint_{\gamma} (x^2 + y^2)\, ds \qquad \text{(ii)} \oint_{\gamma} (-y, x) \cdot d(x,y)$$

(Question from a Real Analysis Exam University of Ioannina , Greece)

107. (a) Let $\mathbb{D} \subset \mathbb{R}^2$ be the unit disk and $\partial\mathbb{D}$ be its positive oriented boundary. Evaluate the following line integral

$$\oint_{\partial\mathbb{D}} \left(x - y^3, x^3 - y^2\right) \cdot d(x,y)$$

(b) Can you deduce if the function

$$f(x,y) = \left(x - y^3, x^3 - y^2\right)$$

is a vector field by basing your reasoning **solely** on question **(a)** ?

(Question from a Real Analysis Exam University of Ioannina , Greece)

108. Prove that for every $c > 0$ the set

$$\mathcal{B}_{f,g} = \{(x,y,z) \in \mathbb{R}^3 : (x - f(z))^2 + (y - g(z))^2 \leqslant c \ , \ z \in [a,b]\}$$

has the same volume for every function $f, g : [a,b] \to \mathbb{R}$.

109. Consider the subset of $\mathbb{R}^3$

$$\mathcal{B} = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 \leqslant z \leqslant a\}, \ a > 0$$

(a) Evaluate

(i) the volume of $\mathcal{B}$.

(ii) the triple integral

$$\mathcal{T} = \iiint_{B} (x^2 + y^2) z \, d(x,y,z)$$

(iii) the area of the boundary of $\mathcal{B}$.

(iv) the surface integral

$$\mathcal{S} = \oint_{\partial B} \sqrt{1 + 4z^2}\, d\sigma$$

(b) Express the volume of $\mathcal{B}$ through a suitable continuously differentiable $G : \mathbb{R}^3 \to \mathbb{R}^3$ and through a suitable surface integral.

110. Prove that the work

$$\mathcal{W} = -\oint_{\gamma} \frac{(x,y,z)}{\left(x^2 + y^2 + z^2\right)^{3/2}} \cdot d(x,y,z)$$

produced along a $\mathcal{C}^1$ oriented curve γ of $\mathbb{R}^3 \setminus \{(0,0,0)\}$ depends only on the distances of starting and ending point of γ about the origin.

111. Let $\mathcal{V}_n(R)$ be the volume of the ball of center 0 and radius $R > 0$ in $\mathbb{R}^n$. Prove that for $n \geqslant 3$ it holds that

$$\mathcal{V}_n(1) = \frac{2\pi}{n} \mathcal{V}_{n-2}(1)$$

112. Let $\mathbb{S}$ denote the area bounded by the curves $x^2 y = 1$ and $x^2 y = 2$ as well as the lines $y = x$ and $y = 2x$ and let γ denote its negative oriented boundary. Evaluate

[1] The $a = 1$ case can be interpreted as (the appropriate constant multiple of) the density of a multivariate normal distribution.

[2] Actually the above inequality is a consequence of a stronger one namely this:

$$\psi^{(m)}(z)\psi^{(n)}(z) \geqslant \psi^{\left(\frac{m+n}{2}\right)}(z)$$

whenever $\frac{m+n}{2} \in \mathbb{N}$. The proof of it may be found at Joy of Mathematics.

$$\mathcal{J}=\oint\limits_{\gamma}\left(e^{-x^2}-6y\right)\,dx+\left(4x-7y^7\right)\,dy$$

113. Let $u:\mathbb{R}^2\rightarrow\mathbb{R}$ be a continously differentiable function and let $\mathcal{C}_r$ be the circle of origin $(0,0)$ and radius $r>0$. Prove that:

$$\frac{1}{2\pi}\lim_{r\rightarrow 0}\frac{1}{r}\oint\limits_{\mathcal{C}_r}u\,ds=u(0,0)$$

114. Let $f(\mathbf{x})=\mathbf{x}^\top Q\mathbf{x}$ where $\mathbf{x}^\top=(x_1,\ldots,x_n)\in\mathbb{R}^n$ and Q is the diagonal matrix

$$Q=\begin{pmatrix} q_1 & 0 & \ldots & 0\\ 0 & q_2 & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & q_n\end{pmatrix}\qquad q_i\in\mathbb{R},\ i=1,\ldots,n$$

(a) Give the derivative as well as the Hessian matrix of f.

(b) Give conditions for the q_i such that f has **a)** a local maximum **b)** a local minumum and **c)** neither of the previous ones.

(c) Compute the Taylor polynomial of degree k of f around $\mathbf{x}=\mathbf{0}$ forall $k\in\mathbb{N}$.

115. Let $\mathcal{S}=[0,1]\times[0,1]\subset\mathbb{R}^2$. Evaluate the integral

$$\mathcal{J}=\iint\limits_{\mathcal{S}}\max\{x,y\}\,d(x,y)$$

Hint: It holds that

$$\max\{x,y\}=\begin{cases} x & , \quad 0\leqslant y\leqslant x\leqslant 1\\ y & , \quad 0\leqslant x\leqslant y\leqslant 1\end{cases}$$

Hence

$$\begin{aligned}\int_0^1\int_0^1\max\{x,y\}\,d(x,y)&=\int_0^1\int_0^x x\,d(y,x)+\\ &\quad+\int_0^1\int_0^y y\,d(x,y)\\ &=2\int_0^1\int_0^x x\,d(y,x)\\ &=2\int_0^1 x^2\,dx\\ &=\frac{2}{3}\end{aligned}$$

116. Let M be the interesection of the elliptic cylinder $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\leqslant 1$ and the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\leqslant 1\quad a>0,\ b>0,\ c>0$$

For all $n\in\mathbb{N}$ evaluate the integrals

$$I_n=\iiint\limits_{M}\left(a^2b^2-b^2x^2-a^2y^2\right)^{n-\frac{1}{2}}d(x,y,z)$$

(Question from a Real Analysis Exam University of Ioannina, Greece)

117. Let $\mathcal{C}=[0,1]\times[0,1]\times\cdots\times[0,1]\subseteq\mathbb{R}^n$ be the unit cube. Define the function

$$f(x_1,x_2,\ldots,x_n)=\frac{x_1x_2\cdots x_n}{x_1^{a_1}+x_2^{a_2}+\cdots+x_n^{a_n}}$$

where a_i arbitrary positive constants. For which values of $a_i>0$ is the value of the integral $\int_{\mathcal{C}}f$ finite?

118. Let $f:\mathbb{R}\rightarrow\mathbb{R}$ be a function such that $f(x)\geqslant 0$ and $\int_{-\infty}^{\infty}f(t)\,dt=1$. For $r\geqslant 0$ we define

$$I_n(r)=\int\limits_{x_1^2+x_2^2+\cdots+x_n^2\leqslant r}\!\!\cdots\int f(x_1)f(x_2)\cdots f(x_n)\,d(x_1,x_2,\ldots,x_n)$$

Evaluate $\lim I_n(r)$.

119. Let $\mathbb{D}=\{z\in\mathbb{C}\mid |z|<1\}$. Let $\mathbb{A}$ denote the area measure on $\mathbb{D}$ normalised so that $\mathbb{A}(\mathbb{D})=\pi$. Verify or disprove that

$$\iint\limits_{\mathbb{D}}\left|\log\left(\frac{e}{1-z}\right)\right|^2 d\mathbb{A}=\frac{\pi^3}{6}$$

An interpretation of this integral; if you have two independent uniform $(0,1)$ random variables, the expected value of the maximum is $\frac{2}{3}$. (And the expected value of the minimum is $\frac{1}{3}$.) More generally: if you have n independent uniform $(0,1)$ random variables, the expected value of the maximum is $\frac{n}{n+1}$. In more detail: if you order these random variables after the fact so that $Y_1\leqslant Y_2\leqslant\cdots\leqslant Y_n$, then the expected value of Y_k is $\frac{k}{n+1}$. (The general name for this sort of reasoning is order statistics.)

120. For a given function $f:\mathbb{R}^3\to\mathbb{R}^3$ the $\displaystyle\int_{\mathbb{R}^3}|f(x)|\,dx$ exists. If for every plane $\mathcal{P}$ of $\mathbb{R}^3$ it holds that $\displaystyle\int_{\mathcal{P}} f(x)\,ds=0$ then prove that f is the zero function.

General Topology

121. Find a countable and dense subset of $\mathbb{R}\setminus\mathbb{Q}$ with respect to the usual topology.

122. Let $\mathcal{X}=[0,+\infty)\cup\{+\infty\}$. We endow it with the metric

$$\rho(x,y)=|\arctan x-\arctan y|$$

Prove that under this metric $\mathcal{X}$ is separable , complete and compact.

123. Does there exist an enumeration $\{q_n\in\mathbb{Q}:n\in\mathbb{N}\}$ of $\mathbb{Q}$ such that

$$\mathbb{R}\neq\bigcup_{n=1}^{\infty}\left(q_n-\frac{1}{n},q_n+\frac{1}{n}\right)$$

124. Prove that there does not exist an $1-1$ and continuous mapping from $\mathbb{R}^2$ to $\mathbb{R}$.

125. Let Ω be a metric space. Suppose that every bounded subset of Ω has at least one accumulation point. Prove that Ω is a complete metric space.

126. (a) Let (X,ρ) be a compact metric space and let $f:X\to X$ be an isometry. Prove that f is onto.

(b) Prove that the ℓ^2 space (that is the space of the sequences for which $\sum\limits_{n=1}^{\infty}x_n^2$ converges) is not compact endowed by the metric

$$\rho(x_n,y_n)=\sqrt{\sum_{n=1}^{\infty}(x_n-y_n)^2}$$

127. Prove that there exists no continuous and $1-1$ map (depiction) from a sphere to a proper subset of it.

128. Is the set $\mathcal{S}=\mathbb{R}^2\setminus\mathbb{Q}\times\mathbb{Q}$ complete? Give a brief explanation.

As a hint you may use Fourier transform.

129. Let $\mathbb{R}^+=\{x\in\mathbb{R}:x>0\}$. Endow it with the metric

$$d(x,y)=\left|\frac{1}{x}-\frac{1}{y}\right|$$

(a) Show that the sequence $a_n=n$ is a Cauchy one.

(b) Is the sequence $\frac{1}{n}$ a Cauchy one?

(c) Show that any sequence a_n in $\mathbb{R}^+$ converges in $\mathbb{R}^+$ in the metric d above if and only if it converges in $\mathbb{R}$ in the standard metric $|x-y|$ and that the limits in the two cases are equal.

130. Let us define the following function:

$$f(x)=\begin{cases}x & , \quad 0\leqslant x<1\\ 1 & , \quad x>1\end{cases}$$

as well as $d_m(x,y)=f(|x-y|)$.

(a) Show that d_m is a metric on $\mathbb{R}$. *You may call it the mole metric. If points are close (closer than one meter), their distance is the usual one, but are they far apart (more than one meter) we do not distinguish between their distances; they are just far apart.*

(b) Show that $\mathbb{R}$ endowed with the above metric is complete and bounded but not compact. Is it totally bounded? Why / Why not?

131. Prove that the set $\mathbb{R}^2\setminus\{0,0\}$ is not simply connected.

132. Find a sequence of open sets $\{G_n\}_{n\in\mathbb{N}}$ of $\mathbb{R}$ such that

$$\mathbb{Z}=\bigcap_{n=1}^{\infty}G_n$$

Well , the problem actually is not of an analysis nature but that of Algebraic Topology. Try to construct a deformation retraction from $\mathbb{R}^2\setminus\{0,0\}$ to $\mathbb{S}^1$ (the unit circle). For example take $f(x)=\dfrac{x}{\|x\|}$. Then the fundamental groups are isomorphic , however $\pi_1(\mathbb{S}^1)\cong\mathbb{Z}$ and hence the fundamental group is not trivial. Therefore , the set is not simply connected.

Simply take

$$G_n=\bigcup_{m\in\mathbb{Z}}\left(m-\frac{1}{n},m+\frac{1}{n}\right)$$

133. (a) Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Prove that the set

$$\mathcal{D}(\theta) = \left\{ (\cos 2n\pi\theta, \sin 2n\pi\theta) \in \mathbb{R}^2 : n \in \mathbb{N} \right\}$$

is a dense subset of the circle $\mathbb{S}^1 : x^2 + y^2 = 1$.

(b) Find a countable and dense subset of $\mathbb{R} \setminus \mathbb{Q}$ with respect to the usual metric.

134. Let us denote $\mathbb{S}^2$ the unit sphere that is the set

$$\mathbb{S}^2 = \left\{ (x, y, z) \in \mathbb{R}^3 \middle| x^2 + y^2 + z^2 = 1 \right\}$$

If $f : \mathbb{S}^2 \to \mathbb{S}^2$ is a continuous function such that $f(x) \neq f(-x)$ forall $x \in \mathbb{S}^2$ then prove that f is onto.

Integrals and Series

135. Evaluate

$$\mathcal{J} = \int_1^\infty \sum_{n=0}^\infty \frac{-dx}{(n+x)^3}$$

136. Let $a \geqslant -1$. Evaluate

$$\mathcal{J} = \int_0^{\pi/2} \log\left(1 + a \sin^2 x\right) \, dx$$

137. Let $n \in \mathbb{N} \mid n > 2$. Prove that

$$\int_0^\infty \frac{\log\left(\frac{1}{x}\right)}{(1+x)^n} \, dx = \frac{1}{n-1} \sum_{k=1}^{n-2} \frac{1}{k}$$

138. Evaluate the integral

$$\mathcal{J} = \int_0^1 \frac{\arctan \dfrac{x}{x+1}}{\arctan \dfrac{1+2x-2x^2}{2}} \, dx$$

(Russian Mathematical Olympiad)

139. For any positive integer n , let $\langle n \rangle$ denote the closest integer to $\sqrt{n}$. Evaluate the sum

$$\mathcal{S} = \sum_{n=1}^\infty \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n}$$

(Putnam 2001)

140. Prove that

$$\int_0^1 \prod_{n=1}^\infty (1 - x^n) \, dx = \frac{4\pi\sqrt{3} \sinh \frac{\pi\sqrt{23}}{3}}{\sqrt{23} \cosh \frac{\pi\sqrt{23}}{2}}$$

141. Evaluate the integral

$$\mathcal{J} = \int_0^1 \frac{\arctan \sqrt{2+x^2}}{(1+x^2)\sqrt{2+x^2}} \, dx$$

142. Let $a \in \mathbb{R}$. Evaluate the integral

$$\mathcal{J} = \int_{-\infty}^\infty \frac{\cos ax}{e^x + e^{-x}} \, dx$$

143. Evaluate the integral

$$\mathcal{J} = \int_0^\infty \frac{x^2 - 4}{x^2 + 4} \frac{\sin 2x}{x} \, dx$$

144. Evaluate the double series

$$\mathcal{S} = \sum_{k=1}^\infty \frac{(-1)^{k-1}}{k} \sum_{n=0}^\infty \frac{1}{k2^n + 1}$$

(Putnam 2016)

145. Evaluate the integral

$$\mathcal{J} = \int_0^1 \left(\frac{1}{1-x} + \frac{1}{\ln x} \right) dx$$

This integral is known with the name "Ahmed's integral" .

The evaluation of this integral allows to tell that

$$\mathfrak{Re}\left[\psi^{(0)}\left(\frac{3}{4} - \frac{ia}{4}\right) - \psi^{(0)}\left(\frac{1}{4} - \frac{ia}{4}\right) \right] = \pi \mathrm{sech}\left(\frac{\pi a}{2}\right)$$

where $\psi^{(0)}$ is the digamma function.

One can also evaluate the general form

$$\int_0^1 \left(\frac{1}{1-x} + \frac{1}{\ln x} \right)^m dx \quad m \geqslant 1$$

146. Let $\psi^{(1)}$ denote the trigamma function. Evaluate the sum

$$\mathcal{S} = \sum_{n=1}^{\infty} (-1)^{n-1} \left(\psi^{(1)}(n)\right)^2$$

(Cornel Ioan Valean)

147. Let Li_2 denote the dilogarithm function and Γ denote the Gamma function. Prove that

$$\int_0^1 \left(\mathrm{Li}_2\left(e^{-2\pi i x}\right) + \mathrm{Li}_2\left(e^{2\pi i x}\right)\right) \log\Gamma(x)\,dx = \frac{\zeta(3)}{2}$$

where ζ is the Riemann zeta function.

148. Let Li_2 denote the dilogarithm function. Prove that

$$\int_0^{\infty} \mathrm{Li}_2\left(e^{-\pi x}\right) \arctan x\,dx = \frac{\pi^2}{18} - \frac{3\zeta(3))}{8}$$

149. Prove that

$$\sum_{n=1}^{\infty} \arctan\left(\frac{10n}{(3n^2+2)(9n^2-1)}\right) = \ln 3 - \frac{\pi}{4}$$

150. Let ζ denote the Riemann zeta function. Prove that

$$\sum_{k=1}^{\infty} \frac{k\zeta(2k)}{4^{k-1}} = \frac{\pi^2}{4}$$

151. Let Li_3 denote the trilogarithm function. Prove that

$$\sum_{n=1}^{\infty} \mathrm{Li}_3\left(e^{-2n\pi}\right) = \frac{7\pi^3}{360} - \frac{\zeta(3)}{2}$$

(Seraphim Tsipelis)

152. Prove that

$$\int_0^{2-\sqrt{3}} \frac{\arctan t}{t}\,dt = \frac{\pi}{12}\log\left(2-\sqrt{3}\right) + \frac{2\mathcal{G}}{3}$$

where $\mathcal{G}$ denotes the Catalan constant.

153. Prove that

$$\sum_{n=1}^{\infty} \frac{\zeta(2n+1)}{(n+1)(2n+1)} = 1 - \gamma$$

where γ stands for the Euler - Mascheroni constant.

(Seraphim Tsipelis, Kotronis Anastasios)

154. Evaluate the following double series

$$\mathcal{S} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} (-1)^{m+n} \frac{m\ln(m+n)}{(m+n)^3}$$

(Enkel Hysnelaj)

155. Let $\mathcal{H}_n$ denote the n-th harmonic number. Prove that

$$\sum_{n=1}^{\infty} \frac{\mathcal{H}_n}{n}\left(\zeta(2) - \sum_{k=1}^{n}\frac{1}{k^2}\right) = \frac{7\zeta(4)}{4}$$

where ζ is the Riemann zeta function.

156. Let $\mathcal{H}_n$ denote the n-th harmonic number. Prove that

$$\sum_{n=1}^{\infty} \frac{\mathcal{H}_n}{n}\cos\left(\frac{n\pi}{3}\right) = -\frac{\pi^2}{36}$$

157. Prove that

$$\sum_{j=2}^{\infty}\prod_{k=1}^{j} \frac{2k}{j+k-1} = \pi$$

158. *This series may be called as "The harmony of the harmony"* . Evaluate the series

$$\sum_{n=1}^{\infty} \frac{1}{(n+2)2^{n+2}} \sum_{k=1}^{n}\frac{1}{k+1}\sum_{m=1}^{k}\frac{1}{m} = \frac{\ln^3 2}{6}$$

159. Let $\mathbb{Z} \ni k \geqslant 1$. Prove that

$$\int_0^1 \ln^k(1-x)\ln x\,dx = (-1)^{k+1}k!\left(k+1-\sum_{m=2}^{k+1}\zeta(k)\right)$$

where ζ denotes the Riemann zeta function.

(Ovidiu Furdui)

160. Evaluate the series

$$\mathcal{S} = \sum_{n=1}^{\infty} \frac{\cos \frac{n\pi}{3}}{9 - 4n^2}$$

161. Let $r \in \mathbb{R}$. Prove that

$$\sum_{n=-\infty}^{\infty} \arctan\left(\frac{\sinh r}{\cosh n}\right) = \pi r$$

(H. Ohtsuka)

162. Evaluate

$$\int_{-\infty}^{\infty} \frac{\arctan x}{x^2 + x + 1} \, dx$$

163. Let Γ denote the Gamma function. Evaluate the integral

$$\int_0^1 \bigg(\log \Gamma(x) + \log \Gamma(1-x)\bigg) \log \Gamma(x) \, dx$$

164. Evaluate the integrals

$$\text{(i)} \quad \int_0^{\infty} \frac{\ln x}{e^x + 1} \, dx \qquad\qquad \text{(ii)} \quad \int_0^{\infty} \frac{\ln x}{e^x - 1} \, dx$$

165. Let erf denote the error function. Prove that

$$\int_0^{\infty} e^{-x} \operatorname{erf}^2(x) \, dx = \frac{2\sqrt{2}}{\pi} \arctan \frac{1}{\sqrt{2}}$$

166. Evaluate

$$\int_0^{\infty} \left(\frac{x}{e^x - e^{-x}} - \frac{1}{2}\right) \frac{dx}{x^2}$$

167. Prove that

$$\int_0^1 \frac{\log(1+x)\log^2 x}{1-x} \, dx = \frac{7}{2} \log 2 \zeta(3) - \frac{19}{720}\pi^4$$

(Cornel Ioan Valean)

168. Let $\mathcal{H}_n$ denote the n-th harmonic number. Evaluate the sum

$$\mathcal{S} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{k+n} \frac{\mathcal{H}_{k+n}^2}{k+n}$$

(Cornel Ioan Valean)

169. Calculate

$$\mathcal{S} = \sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{1 + k \log k}{2 + (k+1)\log(k+1)}$$

170. Calculate

$$\mathcal{S} = \sum_{n=1}^{\infty} \arctan(\sinh n) \arctan\left(\frac{\sinh 1}{\cosh n}\right)$$

(H. Ohtsuka)

171. Let $\{\cdot\}$ denote the fractional part. Evaluate

$$\mathcal{J} = \int_0^{\pi/2} \frac{\{\tan x\}}{\tan x} \, dx$$

172. Calculate

$$\mathcal{J} = \int_0^{\pi/2} x \ln \tan x \, dx$$

173. Let γ denote the Euler - Mascheroni constant. Prove that

$$\int_0^{\infty} \frac{\cos x^2 - \cos x}{x} \, dx = \frac{\gamma}{2}$$

174. Calculate

$$\int_0^{\infty} \frac{\log x}{(2x+1)(x^2+x+1)} \, dx$$

175. Let $\{\cdot\}$ denote the fractional part. Evaluate

$$\int_0^1 \left\{\frac{1}{x}\right\}^2 \left\{\frac{1}{1-x}\right\} dx$$

176. Let Ω denote the root of the equation $xe^x = 1$. Prove that

$$\int_{-\infty}^{\infty} \frac{dx}{(e^x - x)^2 + \pi^2} = \frac{1}{1 + \Omega}$$

The more general identity

$$\prod_{n=-\infty}^{\infty} \left(1 + \frac{\sin r}{\cosh n}\right) = e^{\pi r - r^2}$$

for $\mathfrak{Re}(r) = 0$ seems to be true as pointed out by Tintarn at AoPS.com.

177. Evaluate the series

$$\mathcal{S} = \sum_{n=-\infty}^{\infty} \frac{x^2}{n^2+n-1}$$

as well as the product

$$\Pi = \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2+n-1}\right)$$

178. Let ζ denote the Riemann zeta function. Prove the identity:

$$\frac{1}{2\pi}\mathrm{Li}_2\left(e^{-2\pi}\right) = \log(2\pi) - 1 - \frac{5\pi}{12} - \sum_{n=1}^{\infty} \frac{(-1)^n \zeta(2n)}{n(2n+1)}$$

where Li_2 denotes the dilogarithm function.

179. Let $\mathcal{G}$ denote the Catalan's constant. Prove that

$$\frac{\pi}{2} \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n+1)4^n} = \mathcal{G}$$

where ζ denotes the Riemann zeta function and $\zeta(0) = -\frac{1}{2}$.

180. Let $s \in \mathbb{C}$ such that $\mathfrak{Re}(s) > 1$. Evaluate the following double Euler sum

$$\mathcal{S} = \sum_{(j,k)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(j^2+k^2)^s}$$

181. Evaluate the integral

$$\mathcal{J} = \int_0^{\pi/2} \sin^2 x \log\left(\sin^2(\tan x)\right)\, dx$$

182. Let $0 \leqslant \alpha, \beta \leqslant \pi$ and $\kappa > 0$. Prove that

$$\int_0^{\infty} \frac{1}{x} \log\left(\frac{x^2 + 2\kappa x\cos\beta + \kappa^2}{x^2 + 2\kappa x \cos\alpha + \kappa^2}\right) dx = \alpha^2 - \beta^2$$

183. Let γ denote the Euler – Mascheroni constant. Define $F(x) = \sum\limits_{n=1}^{\infty} x^{2^n}$. Prove that

$$\gamma = 1 - \int_0^1 \frac{F(x)}{1+x}\, dx$$

184. Let $\mathcal{B}_n$ denote the n -th Bernoulli number. Prove that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\mathcal{B}_{2n} x^{2n}}{2n\,(2n)!} = \log\frac{x}{2} - \log\sin\frac{x}{2}$$

185. Evaluate the integral

$$\mathcal{J} = \int_0^1 \frac{1-x}{\log x} \sum_{n=0}^{\infty} x^{2^n}\, dx$$

186. Prove that

$$\sum_{n=1}^{\infty} \frac{\coth n\pi}{n^7} = \frac{19\pi^7}{56700}$$

187. Evaluate the sum

$$\mathcal{S} = \sum_{n=-\infty}^{\infty} \frac{\log\left|n + \frac{1}{4}\right|}{n + \frac{1}{4}}$$

(Seraphim Tsipelis)

188. Let $\mathcal{H}_n$ denote the n-th harmonic sum. Evaluate the sum:

$$\mathcal{S} = \sum_{n=1}^{\infty} \left(\mathcal{H}_n - \log n - \gamma - \frac{1}{2n} + \frac{1}{12n^2}\right)$$

(M. Omarjee)

189. Prove that

$$\prod_{n=0}^{\infty} \left(\prod_{k=0}^{n} (k+1)^{(-1)^{k+1}\binom{n}{k}}\right)^{\frac{n(n+1)}{2^{n+3}}} = e^{7\zeta(3)/24\zeta(2)}$$

where ζ denotes the Riemann zeta function.

190. Let $\mathbb{R} \ni s > 2$. Evaluate the (double) sum:

$$\mathcal{S} = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{m^2+4mn+n^2}{(m^2+mn+n^2)^s}$$

(Kent Merryfield)

191. Let $a \in [-\pi, \pi]$ and let us denote with Ci the Cosine integral function. Evaluate the series

$$\mathcal{S} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\operatorname{Ci}(na)}{n^2}$$

192. Let α , $\beta \in \mathbb{R}$ such that $0 < \alpha < \beta$. Prove that

$$\int_0^{\infty} \frac{\log x}{(x+\alpha)(x+\beta)}\,dx = \frac{1}{2(\beta-\alpha)}\left[\log^2\beta - \log^2\alpha\right]$$

(Grigorios Kostakos)

193. Let γ denote the Euler - Mascheroni constant. Prove that

$$\int_0^{\infty} \frac{\cos x^n - \cos x^{2n}}{x} \log x\,dx = \frac{12\gamma^2 - \pi^2}{2(4n)^2}$$

194. Calculate

$$\mathcal{M} = \int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty} \frac{\prod\limits_{m=1}^{n}\cos(x_m)}{\sum\limits_{m=1}^{n} x_m}\,d(x_1, x_2, \ldots, x_n)$$

195. Let $\mathcal{H}_n$ denote the n -th harmonic number. Prove that $|z| < 1$ it holds that

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}\mathcal{H}_{2k}}{2k+1} z^{2k+1} = \frac{\arctan z}{2}\log\left(1+z^2\right)$$

196. Let $\mathcal{B}_n$ denote the n-th Bernoulli number. Prove that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\mathcal{B}_{2n}x^{2n}}{2n\,(2n)!} = \log\frac{x}{2} - \log\sin\frac{x}{2}$$

197. Let $\mathcal{G}$ denote the Catalan's constant and $\mathcal{H}_n$ the n -th harmonic number. Prove that

$$\sum_{n=1}^{\infty}\left(\frac{\mathcal{H}_{4n-3}}{4n-3} - \frac{\mathcal{H}_{4n-2}}{4n-2}\right) = \frac{\pi^2}{64} + \frac{\pi\log 2}{32} + \frac{\mathcal{G}}{2} - \frac{3\log^2 2}{16} - \frac{3\pi\log 2}{32}$$

(Cornel Ioan Valean)

198. Let $\mathbb{A}$ denote the Glashier - Kinkelin constant and γ the Euler - Mascheroni constant. Prove that

$$\prod_{k=1}^{\infty}\prod_{n=1}^{\infty}\prod_{m=1}^{\infty}(k+n+m)^{\frac{(-1)^{k+m+n}}{k+m+n}} = \frac{\mathbb{A}^{3/2}}{\pi^{3/4}e^{1/8-(7/12+\gamma)\log 2+\frac{1}{2}\log^2 2}}$$

(Cornel Ioan Valean)

199. Prove that

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \cdots\right)^2 = \frac{\pi^2\ln 2}{6} - \frac{\ln^3 2}{3} - \frac{3}{4}\zeta(3)$$

(Ovidiu Furdui)

200. Let k be a positive integer. Evaluate the multiple sum

$$\mathcal{S} = \sum_{i_1,\ldots,i_k \geqslant 1} \frac{1}{i_1\cdots i_k(i_1+\cdots+i_k)^2}$$

(Ovidiu Furdui)

The most straight forward approach is to use Fourier series beginning by equation (2) at the link. The final answer is

$$\mathcal{S} = \frac{\gamma\pi^2}{12} + \frac{\pi^2\ln a}{12} - \frac{\pi^2\ln 2}{12} - \frac{\zeta'(2)}{2} - \frac{a^2}{8}$$

where γ denotes the Euler - Mascheroni constant.

The interested reader might as well give a try the following integral

$$\mathcal{J} = \int_0^{\infty}\frac{\log^2 x}{(x+\alpha)(x+\beta)}\,dx$$

Currently I do not have a solution on this but the most straight forward idea is to actually try to find the number of ways n can be written as a sum of three numbers and reduce the triple product into a single one.

For $k=1$ the sum equals $\dfrac{(k+1)!\zeta(k+2)}{2}$ whereas for $k \geqslant 2$ the sum equals

$$k!\left(\frac{k+1}{2}\zeta(k+2) - \frac{1}{2}\sum_{i=1}^{k-1}\zeta(k+1-i)\zeta(i+1)\right)$$

201. Evaluate

$$\int_0^\infty \int_0^\infty \frac{d(x,y)}{(e^x+e^y)^2}$$

(Ovidiu Furdui)

202. Evaluate the integral

$$\int_0^\infty \frac{e^x-1}{e^x+1} \ln^k \left(\frac{e^x+1}{e^x-1}\right) dx$$

203. Let μ denote the Möbius function. Evaluate the series

$$\mathcal{S} = \sum_{n=1}^\infty \frac{(-1)^{\mu(n)}}{n^s}$$

where $\mathfrak{Re}(s) > 1$.

204. Let $n \in \mathbb{N}$ and ζ denote the Riemann zeta function. Prove that

$$\int_0^{\pi/2} (\log \sin x)^n \tan x \, dx = (-1)^n \frac{n!\, \zeta(n+1)}{2^{n+1}}$$

205. Let $\mathcal{G}$ denote the Catalan's constant. Prove that

$$27 \sum_{n=0}^\infty \frac{16^n}{(2n+3)^3 (2n+1)^2 \binom{2n}{n}^2} = \frac{27}{2}\left(7\zeta(3) + (3-2\mathcal{G})\pi - 12\right)$$

206. Let $\mathcal{H}_n$ denote the n - th harmonic number. Prove that

$$\sum_{n=1}^\infty \frac{(-1)^{n+1}\mathcal{H}_n \mathcal{H}_{n+1}}{(n+1)^2} = \frac{\pi^4}{480}$$

207. Express in terms of dilogarithm the series

$$\mathcal{S} = \sum_{n=1}^\infty (n \operatorname{arccot} n - 1)$$

The above series was proved by Jacopo D' Aurizio , an MSE user. The series goes deeper and is actually a closed form of the hypergeometric function

$${}_4F_3\left(1,1,1,\frac{3}{2};\frac{5}{2},\frac{5}{2},\frac{5}{2};1\right)$$

208. Let lcm denote the least common multiple. Prove that for all $s > 1$ it holds that

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{1}{\operatorname{lcm}^s(m,n)} = \frac{\zeta^3(s)}{\zeta(2s)}$$

where ζ is the Riemann zeta function.

209. The n -th Fibonacci number is defined as $F_0 = 0$, $F_1 = 1$ and recursively via the relation

$$F_{n+2} = F_{n+1} + F_n \quad \text{for all} \quad n \geqslant 0$$

Prove that

$$\sum_{n=0}^\infty \arctan\left(\frac{(-1)^n}{F_{n+1}(F_n + F_{n+2})}\right) = \arctan\left(\sqrt{5}-2\right)$$

210. Let ζ denote the Riemann zeta function and let $\mathbb{N} \ni s \geqslant 2$. Prove that

$$\int_0^1 \operatorname{arctanh}^s(x)\, dx = \frac{2\zeta(s)\,(2^s-2)\,\Gamma(s+1)}{4^s}$$

211. Evaluate the product

$$\Pi = \prod_{n=1}^\infty \left(1+\frac{1}{4n}\right)^2 \left(\frac{2n+1}{2n+1+(-1)^{n-1}}\right)^{(-1)^{n-1}}$$

212. Let T_n denote the n - th triangular number. Evaluate

$$\sum_{n=1}^\infty \frac{1}{(8T_n-3)(8T_{n+1}-3)}$$

213. Let $\psi^{(0)}$ denote the digamma function and μ the Möbius function. Prove that

$$\sum_{n=1}^\infty \frac{\mu(n)}{n}\psi^{(0)}\left(1+\frac{1}{n}\right) = \frac{1}{2}$$

214. Let μ denote the Möbius function. Prove that

$$\sum_{n=1}^\infty \frac{\mu(n)\log n}{n} = -1$$

215. Let gd denote the Gudermannian function.
Evaluate the integral:

$$\mathcal{J} = \iint\limits_{[0,1]^2} \frac{\operatorname{gd}(\log xy)}{1-xy} \, \mathrm{d}(x,y)$$

216. Let F_n denote the n -th Fibonacci number and let $\mathcal{H}_n^{(2)}$ denote the n -th harmonic number of weight 2. Evaluate the series

$$\mathcal{S} = \sum_{n=1}^{\infty} \frac{F_n \mathcal{H}_{n-1}^{(2)}}{n^2 \binom{2n}{n}}$$

217. Let $\psi^{(1)}$ denote the trigamma function. Prove that

$$\sum_{n=1}^{\infty} \psi^{(1)}(n) x^n = \frac{x}{1-x} \left(\zeta(2) - \operatorname{Li}_2(x) \right)$$

In continuity , investigate for which $x \in \mathbb{R}$ does the series converge .

218. Let $\psi^{(1)}$ denote the trigamma function. Prove that

$$\sum_{n=1}^{\infty} \frac{\psi^{(1)}(n) \psi^{(1)}(n+1)}{n^2} = \frac{\pi^6}{840} = \frac{9\zeta(6)}{8}$$

(Seraphim Tsipelis)

219. Let Li_2 denote the dilogarithm function. Evaluate the double integral

$$\mathcal{J} = \int_0^1 \int_0^1 \frac{\log x \log y}{(1-x)(1-y)} \frac{\operatorname{Li}_2(xy)}{xy} \, \mathrm{d}(x,y)$$

220. Evaluate the series

$$\Omega = \sum_{n=1}^{\infty} \arctan \left(\frac{9}{9 + (3n+5)(3n+8)} \right)$$

(Dan Sitaru)

Appendix

In this appendix we shall present some open problems.

1. Can we cover a unit square with $\frac{1}{k} \cdot \frac{1}{k+1}$ rectangles? Here $k \in \mathbb{N}$.

2. Is the sequence $\left(\frac{3}{2}\right)^n \mod 1$ dense in the unit interval?

3. Is it true that

$$\sum_{n=0}^{\infty} \frac{1+14n+76n^2+168n^3}{2^{20n}} \binom{2n}{n}^7 = \frac{32}{\pi^3}$$

4. *(The following is called Giuga Conjecture or Agoh-Giuga Conjecture and its origins can be traced back in* 1950*.)* A positive integer $p > 1$ is prime if and only if

$$\sum_{i=1}^{p-1} i^{p-1} \equiv -1 \pmod p$$

5. Why is it so difficult to prove that $e+\pi$ is irrational?

6. Let $\left(\frac{n}{7}\right)$ denote the Legendre symbol. Is it true that

$$\frac{24}{7\sqrt{7}} \int_{\pi/3}^{\pi/2} \log \left| \frac{\tan t + \sqrt{7}}{\tan t - \sqrt{7}} \right| \, dt = \sum_{n=1}^{\infty} \left(\frac{n}{7}\right) \frac{1}{n^2}$$

7. Is the Catalan's constant defined as

$$\mathcal{G} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$$

irrational?

8. Let $\mathcal{H}_n$ denote the n - th Harmonic number. Is it true that for all $n \geqslant 1$ it holds that

$$\sum_{d \mid n} d \leqslant \mathcal{H}_n + (\log \mathcal{H}_n) e^{\mathcal{H}_n}$$

This kind of identity is amenable in principle to automatic theorem-proving methods, but (using known techniques) is out of reach of current computers. Another such formula is the Cullen's Pi Formula that can be found here.

Actually Jeff Lagarias showed that this is equivalent to the Riemann hypothesis!

9. Let $x_0 = 2$. Is it true that the sequence $\{x_n\}_{n \in \mathbb{N}}$ defined as

$$x_{n+1} = x_n - \frac{1}{x_n}$$

is unbounded?

10. Does the series

$$\mathcal{S} = \sum_{n=1}^{\infty} \frac{1}{n^3 \sin^2 n}$$

converge?

11. Is it true that

$$\lim_{n \to +\infty} \frac{1}{n^2 \sin n} = 0$$

12. Let p_n denote the n -th prime. Is the series

$$\mathcal{S} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{p_n}$$

convergent?

13. Is there a dense subset of a plane having only rational distances between its points?

14. For every odd prime is it true that one has

$$0! + 1! + \cdots + (p-1)! \not\equiv 0 \pmod p$$

15. *(The following is known as Littlewood's conjecture.)* For α , $\beta \in \mathbb{R}$ is it true that

$$\liminf_{n \to +\infty} (n \cdot \|n\alpha\| \cdot \|n\beta\|) = 0$$

Here $\|\cdot\|$ denotes the distance to the nearest integer.

16. What is the largest possible volume of the convex hull of a space curve having unit length?

We would expect this to tend to zero, but the proof is beyond what is currently known. It is expected that the irrationality measure of π is 2 (it is known that all but a zero-measure set of real numbers have irrationality measure 2). Therefore, it is expected that the sequence tends to 0 but currently there is no proof for that.

The origin of this problem traces back to Paul Erdős .

This is known as Kurepa's conjecture. A proof was claimed and published in 2004 but the claim was withdrawn in 2011.

References

Here is a list of references that indicate , potentially , the source of the majority of the problems or that of the appendix.

International Fora

- Mathematics Stack Exchange

  Description: Mathematics Stack Exchange is a Q&A site that allows users to ask and answer questions. It is quite rich in interesting questions of all levels from trivial up to very challenging ones.

- Art of Problem Solving

  Description: Art of Problem Solving (abbrev: AoPS) is a site that is a great resource of mathematical competitions. It also has a college forum with plenty of interesting questions and answers.

- mathimatikoi.org/forum

  Description: mathimatikoi.org (from the greek word that means mathematicians) is an English forum of university mathematics. Its main focus is in college level mathematics and some branches of Euclidean Geometry.

- Integrals and Series

  Description: Integrals and Series is a forum on discussion on Integrals and Series only. It has many topics on the evaluation of challenging integrals and series as well as studies on special functions.

  Note: *This site / forum is using Tapatalk and MathJaX is no longer rendering math equations. You are* ***strongly*** *adviced to use a bookmark so that it renders MathJaX. Unfortunately , this site (which once was a valuable resource of integrals and series) is useless anymore.*

Local Fora

- mathematica.gr

  Description: mathematica.gr is a greek site on mathematical discussions. It is a great resource on mathematical competitions , mathematical news, teaching technics as well as university and applied mathematics.

Other Sites

- tolaso.com.gr

  Description: The editor's personal site.

Institutions

- University of Ioannina, Ioannina, Greece
- University of Athens , Athens, Greece
- University of Wisconsin , USA
- University of Michigan, Michigan , USA

Books / Journals

- American Mathematical Monthly
- Romanian Mathematical Monthly
- Asymmetry
- Rudin W. Principals of Mathematical Analysis
- Principals of Multivariable Calculus , Giannoulis Ioannis , University of Ioannina
- Complex Analysis , Stein E.M and Shakarchi R

Other References

These other references may include facebook groups.